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AMPLITUDE INSTABILITIES IN SYSTEMS OF NONLINEAR COUPLED OSCILLATORS

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AMPLITUDE INSTABILITIES IN SYSTEMS OF NONLINEAR COUPLED OSCILLATORS

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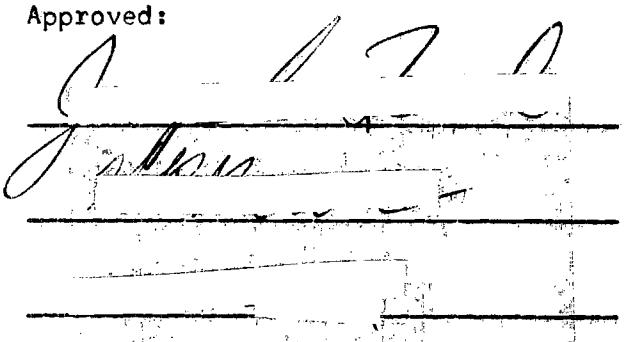

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SUMMARY

Systems of coupled oscillators are used as the basic model in the theoretical investigations for such diverse fields as thermodynamics and astronomy. In many cases the harmonic or linear approximation is sufficient to provide an adequate description of the problem at hand, but some important physical phenomena, such as the approach to equilibrium, cannot be described unless nonlinear forces are included in the model. For systems consisting of a finite number of linearly coupled oscillators the solutions of the equations of motion are available in principle and the general mathematical theory of such systems is extensive. On the other hand, there is no comparable theory available for systems of coupled nonlinear oscillators, and each problem must be handled individually.

Since the general system of n coupled nonlinear oscillators is hopelessly complicated, we shall consider in this study only the most simple case of interest, namely that of a conservative system of two coupled oscillators. Such a system possesses at least one integral of the motion, the total energy, and questions of considerable interest hinge on knowing whether or not there is an additional well-behaved integral. For this additional integral to be physically interesting, it is necessary that it be sufficiently restrictive of the system's motion so that a knowledge of the integral enables one to make deductions as to the nature of the system motion. Integrals of this type are called "isolating integrals." If an additional isolating integral, often called the "third" integral, exists, the system phase point executes conditionally periodic motion on tori in the four-dimensional phase space of the system.

There are a number of theoretical results concerning the third integral, but most of these results are negative in that under certain general conditions they show the nonexistence of an additional isolating integral analytic in the system parameters or the nonconvergence of certain methods for calculating this integral. The relevance of these theorems to the qualitative study of the motion of conservative dynamical systems is not clear, and in light of these theoretical uncertainties, a number of investigators have attempted numerical studies of specific systems. Recent significant work along these lines is due to Henon and Heiles, who studied a Hamiltonian of the form

$$H = \frac{p_1^2 + x_1^2}{2} + \frac{p_2^2 + x_2^2}{2} + x_1^2 x_2 - \frac{1}{3} x_2^3 \,. \qquad (1)$$

where x_1, x_2 are coordinates and p_1, p_2 are the corresponding conjugate momenta. The technique used in their studies is based on the following observations. It is possible to solve the energy integral for one of the variables, say p_1. Since $p_1^2 \geq 0$, it may be seen from Equation (1) that if there are no other isolating integrals, then the remaining three variables range more or less uniformly over a bounded volume in (x_1, x_2, p_2) space. On the other hand the existence of an additional integral means that x_1, x_2, and p_2 are no longer independent; as the system phase point moves along a given trajectory in phase space, the corresponding values of x_1, x_2, and p_2 must lie on a surface in (x_1, x_2, p_2) space. In particular, if one looks at points for which $x_1 = 0$, then a set of points in the (x_2, p_2) plane is obtained, all of which lie on a smooth closed curve (level curve). It may happen that there are no points for which $x_1 = 0$ or only a finite number

of such points. This would be the case if the energy exceeds the dissociation energy. In these instances surfaces of section other than $x_1 = 0$ must be used. However, if there is no third integral, then the points in the (x_2, p_2) plane should be scattered about almost at random. Henon and Heiles found that for Hamiltonian (1) the calculated points all lay on smooth closed level curves for low energies while at higher energies level curves persisted for some initial conditions but not for others. Thus these numerical experiments yielded the puzzling and unexpected result that the third integral is isolating at low energies and at best only partially isolating at higher energies.

There are a number of formal methods for calculating power series expansions for the third integral, and for Hamiltonian (1) perhaps the most tractable method is the one due to Birkhoff. Birkhoff considered the case for which, in the linear approximation, the uncoupled angular frequencies were incommensurate. The anharmonic terms were treated as perturbations on the linear system. A sequence of canonical transformations was used to eliminate as many terms as possible in the Hamiltonian. The result is a Hamiltonian in "normal form" which, if expressed in terms of action-angle variables, J_i, φ_i, may be written

$$H = H(J_1, J_2) \ . \tag{2}$$

The angle coordinates are ignorable and the action variables are therefore constants of the motion. Using the inverses of the transformations from the normalization process, it is possible to express one of the constant action variables as a power series in the original coordinates. This formal power series represents an isolating third integral. However, if

the angular frequencies in the linear approximation are commensurate, the Birkhoff process must be modified. We develop the necessary modification to the Birkhoff method which, if applied to Hamiltonian (1), yields a Hamiltonian of the form

$$H = H(J_1, J_2, \varphi_1 - \varphi_2) \ , \tag{3}$$

which will also be called a normal form.

This Hamiltonian has an additional integral I given by

$$I = J_1 + J_2 \ , \tag{4}$$

as may be verified by calculating the time rate of change of I and then using the equations of motion. Again, by application of the inverses of the transformations used in obtaining the normal form, the integral I can be expressed as a formal power series in the original coordinates.

There is a large amount of lengthy algebraic work involved in reducing a Hamiltonian to normal form, and a series of computer programs has been developed to handle the algebraic formalism. We use these programs to reduce Hamiltonian (1) to normal form through eighth order and to find a power series expansion of the integral I through eighth order in the original coordinates. By using this truncated integral and the Hamiltonian, it is straightforward to calculate numerical approximations to the level curves for Hamiltonian (1). We do this for several energies obtaining excellent agreement at low energies with the level curves obtained by direct integration of the equations of motion. At higher energies the theoretical level curves, that is, the level curves calculated using the truncated integral and the Hamiltonian, deviate from the directly

calculated ones in that the latter begin to exhibit ergodic properties. In particular, the theoretical level curves do not yield the so-called island chains prominent in the directly calculated level curves at higher energies. An island chain is a set of disjoint, closed level curves all of which arise from a single orbit. The central points of an island chain represent a stable periodic solution with a long period.

The level curves of a Hamiltonian in the normal form of Equation (3) cannot exhibit an island chain since all the stable periodic solutions of this type of system are such that the frequencies of all four variables (two coordinates and two momenta) are equal. This implies that the terms in the perturbation which cause these island chains are improperly eliminated in that they perhaps cause a divergence of the normalization method.

We identify these crucial terms as follows. If the Hamiltonian H_o is a function of the action variables, J_1, J_2, only, then J_1 and J_2 are constant and the angle variables increase with constant angular velocities Ω_1 and Ω_2 given by

$$\varphi_1^{\,o} = \frac{\partial H_o}{\partial J_1} = \Omega_1(J_1, J_2) \ , \tag{5a}$$

$$\varphi_2^{\,o} = \frac{\partial H_o}{\partial J_2} = \Omega_2(J_1, J_2) \ , \tag{5b}$$

The level curves of this system are concentric circles centered on the origin. It is found that the addition of a perturbation of the form $f(J_1, J_2) \cos(m\varphi_1 - n\varphi_2)$ to H_o only slightly distorts the original, circular level curves except in the neighborhood of the level curve representing

the torus for which $m\Omega_1 = n\Omega_2$. In this region the level curves show a chain of islands except for the case m = n. The boundary between the island chain and the slightly distorted level curves is a level curve called the separatrix which represents an asymptotic motion of the system. Perturbations proportional to $\cos(m\varphi_1 - n\varphi_2)$ will be called an "m-n resonance," and the region interior to the surface in phase space represented by the separatrix will be called an "m-n resonance zone." It is these resonance terms that are improperly eliminated by the Birkhoff normalization process.

In order to examine the role played by resonances in causing ergodic behavior, three examples are considered. H_o is taken to be

$$H_o = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2. \tag{6}$$

First the effects of a perturbation of the form $cJ_1J_2 \cos 2(\varphi_1 - \varphi_2)$ on Hamiltonian (6) are studied where c is a constant. This Hamiltonian is in normal form and the level curves may be found algebraically. There are two crescent-shaped regions bounded by a separatrix near the $\Omega_1 = \Omega_2$ torus for Hamiltonian (6). There are no island chains, however. Next, Hamiltonian (6) is perturbed by $\beta J_1^{3/2} J_2 \cos(3\varphi_1 - 2\varphi_2)$ where β is a constant. This system is also integrable, having the additional isolating integral $\bar{I} = 2J_1 + 3J_2$. There is a chain of three islands in the level curve diagram in the $x_1 - p_1$ plane near the $3\Omega_1 = 2\Omega_2$ torus for Hamiltonian (6) yielding an integrable system which exhibits an island chain near the $2\Omega_1 = 3\Omega_2$ torus for Hamiltonian (6). In all cases the positions of the periodic solutions and the separatrices have been studied numerically as

functions of energy. It is possible to generalize these results to account for the effects of a general m-n resonance acting on Hamiltonian (2).

Most Hamiltonians, however, have, in addition to terms involving only the action, nonresonant angle-dependent terms and usually more than one resonance term. We do not consider the effects of nonresonant angle-dependent terms; instead we study two examples of Hamiltonians possessing more than one resonant term in addition to the terms involving only the action variables. Such systems are termed "multiply resonant." There are no longer any simple third integrals for these systems such as those for the case of one resonance, and the algebraic results mentioned in the last paragraph are no longer available. Instead we must rely upon the important work on dynamical systems by Kolmogorov. As was mentioned earlier the system phase point for Hamiltonian (2) executes conditionally periodic motion on the surfaces of tori in four-dimensional phase space. Kolmogorov was able to show that for sufficiently small perturbations the majority of these tori are only slightly deformed by the application of the perturbation. However, between the tori that are preserved, there are regions where the tori having originally commensurate frequencies are destroyed. These regions are called "zones of instability." These results suggest that, for a multiply resonant Hamiltonian, the level curves should be regular so long as the resonance zones of the individual resonances are widely separated. The zones of instability are of course always present but are small and difficult to detect on a computer. On the other hand, if the resonance zones overlap, then the zones of instability become macroscopic, and severe breakdown in the level curve structure is the result. This affords an easy way to estimate the region and energy

for which large scale breakdown starts. One simply allows the resonances to act individually upon the unperturbed Hamiltonian and finds algebraically the energy for which overlap occurs. This energy should approximate the start of the breakdown observed by Henon and Heiles.

As a test of these ideas we find the level curves of the Hamiltonian

$$H = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2 + \alpha J_1 J_2 \cos 2(\varphi_1 - \varphi_2) + \beta J_1 J_2^{3/2} \cdot \cos(2\varphi_1 - 3\varphi_1) \quad (7)$$

with $\alpha = \beta = 0.02$. The energy for which overlap starts is found to be 0.2095. At E = 0.18 the level curves are regular, while at 0.2095 there is indeed a considerable degree of breakdown. As a further test, we calculate the level curves for E = 0.20, and at this point the problem of secondary resonances arises; that is, resonances which do not occur explicitly in Hamiltonian (7) but nevertheless exist because of the interaction of the 2-3 resonance with the 2-2 resonance. The secondary resonances can have important effects upon the level curve structure, as will be shown in several examples. It is a straightforward matter to identify the secondary resonances in Hamiltonian (7) by means of the Birkhoff normalization process. Elimination of the 2-3 resonance produces a 4-5 secondary resonance. This 4-5 resonance acts on tori in the neighborhood of the $4\Omega_1 = 5\Omega_2$ torus for Hamiltonian (6), producing a chain of islands. This chain of islands is small but observable on a computer since we know to a good degree of approximation where it should be. At E = 0.20 the 4-5 resonance overlaps with the 2-2 resonance zone and some minor breakdown should be visible. We verify that this is the case by directly finding

the level curves by numerical integration of the equations of motion. It is now seen that the problem of predicting breakdown becomes one of knowing the accuracy of the computer calculated level curves. There are, in general, an infinite number of secondary resonances, each producing an island chain, and the level curve structure is complicated at all energies. Only the relatively low order secondary resonances produce effects large enough to be readily found on a computer, and out method of finding resonance zones and regions of overlap enables one to predict the behavior of the level curves.

It is of interest to further test these ideas for larger perturbations. To do this, we use the Hamiltonian

$$H = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2 + \alpha J_1 J_2 \cos 2(\varphi_1 - \varphi_2) + \beta J_1^{3/2} J_2 \cdot \cos(3\varphi_1 - 2\varphi_2), \quad (8)$$

with $\alpha = .95$ and $\beta = .25$. The 2-2 and 3-2 resonance zones overlap at $E = 0.1210$. However, there is an important 5-4 secondary resonance which causes breakdown to become visible at the much lower energy of $E = 0.08$. The effects of even higher order secondary resonances are apparent for this Hamiltonian.

Finally, we return to the Henon-Heiles system and formulate a method whereby one could in principle predict the appearance of macroscopic breakdown in the level curves of Hamiltonian (1) by means of the methods used in the study of Hamiltonian (7) and (8).

CHAPTER I

INTRODUCTION

Conservative systems of coupled oscillators have long been of interest in physics. Such systems provide models for many physical problems with applications ranging from astronomy to solid state physics (Waters and Ford 1966). In the analysis of oscillator systems, one usually starts with the harmonic or linear approximation. Physically this may be viewed as a set of masses connected together by springs obeying Hooke's law. The body of mathematical theory concerning such systems is very extensive, although intractable in many specific problems. Linear systems are very useful, but they must be regarded as only a "zeroth order" approximation to most physical problems. For example, a large isolated assembly of linearly coupled oscillators can never approach equilibrium in the usual sense because of the existence of normal modes with constant energy in each mode. Thus it is not possible to study the approach to equilibrium of the normal-mode energies for an insulating crystal using linear models alone.

To give a more realistic description of physical phenomena, small non-linear terms are usually added to the coupling between the oscillators. There are indications that this is sufficient to give a qualitative and perhaps even a quantitative description of many interesting physical processes such as the approach to equilibrium, thermal conductivity, etc. (Ziman 1960). Speaking somewhat loosely, the supposed essential effect

of the nonlinearity is to allow the oscillators, as they move, to assume nearly all the positions and velocities allowed by the conservation of energy, i.e. nonlinearity provides ergodic motion. Despite intensive study over the past half century, no one has been able to demonstrate conclusively that nonlinearity does indeed have this essential effect for systems of physical interest. However, recent work by several investigators (Henon and Heiles 1964, Barbanis 1965, Gustavson 1966, Perek 1966) has indicated that nonlinear oscillators may indeed have the desired properties provided that the amplitudes of the motion exceed a certain critical value. The following thesis describes an investigation of this amplitude instability. Since the usefulness of nonlinear models for physical systems is presumed to depend primarily on the ergodic character of the motion rather than on details of the solutions, we now turn to a description of how one determines whether or not the motion has an ergodic character.

The general dynamical system with n degrees of freedom seems hopelessly complicated. We shall therefore, as have previous investigators, restrict our attention to a type of system which exhibits many of the complexities of the higher-dimensional cases yet is simple enough to be somewhat amenable to analysis - the conservative system with two degrees of freedom. For such systems the Hamiltonian is a function of two coordinates, x_1 and x_2, and two momenta, p_1 and p_2. The time evolution of the systems governed by Hamiltonian's equations:

$$\dot{x}_i = \frac{\partial H}{\partial p_i} , \qquad \text{(I-1a)}$$

$$\dot{p}_i = - \frac{\partial H}{\partial x_i} ; \quad i = 1,2 . \qquad \text{(I-1b)}$$

Geometrically the motion of the system may be viewed as the motion of a representative point in a four-dimensional Euclidean phase space. It will be assumed throughout this work that the Hamiltonian is sufficiently smooth that the existence and uniqueness theorems of differential equations apply to Equation (I-1). Because of these theorems there will be only one orbit through each point in phase space.

The integration of the equations of motion is, of course, the central problem in mechanics. Although we shall later have need to integrate the equations of motion, for our purposes exact knowledge of the solutions, as has been mentioned, is not as useful as certain types of qualitative knowledge, such as the existence and stability of periodic solutions, ergodicity, etc. This approach has been shown in the past to be extremely fruitful, and there is a growing literature on these so-called topological methods (Minorsky 1947). A crucial topic in all of this work and one of greatest interest for this study is the existence and character of the integrals of motion since, as we shall see, they so easily provide one with a knowledge of the character of the motion. The Hamiltonian itself is one integral, the energy. The energy integral restricts the system trajectory (in the four-dimensional phase space) to three-dimensional surfaces of constant energy. In some problems there are other restricting integrals, for example, the classical integrals of linear and angular momentum. We may verify however that these integrals are not applicable to Equation (I-1). The question then arises as to whether or not other useful integrals exist. By the term useful, we shall mean an integral that restricts the system motion to a certain hypersurface in much the same fashion as does the energy integral. The terms "uniform" integrals or "isolating" integrals are often

used in this connection. On the other hand a nonisolating integral $I(x_1,p_1,x_2,p_2)$ is such that the solutions of $I(x_1,p_1,x_2,p_2)$ = constant consist of an infinity of sheets which fill the phase space in such a way that the integral does not restrict the motion. Simple examples of such nonisolating integrals are known (Wintner 1941). Moreover the results of several investigations have shown that under fairly general circumstances no additional isolating integrals exist which are well-behaved functions of the system parameters. For example, Poincare has shown that if the Hamiltonian is of the form $H_o + \lambda V_o$, where H_o and V_o satisfy certain weak conditions, and λ is a constant, then there exist no additional isolating integrals analytic in the coupling parameter λ. The relevance of such results for physical problems is not clear, as there may exist isolating integrals which are not analytic in the coupling constant λ (Cherry 1925).

Of greatest interest to us is the existence or nonexistence of an additional isolating integral, termed the "third" integral in some of the recent literature, for a system with two degrees of freedom. As previously remarked, the existence of the energy integral implies that the four-dimensional phase space is divided into invariant three-dimensional manifolds (hypersurfaces of constant energy). If there are no further isolating integrals, then it is to be expected that the system phase point will pass arbitrarily near each point on the constant energy surface, that is, the motion is ergodic and the oscillators may assume nearly all positions and velocities allowed by the energy integral. On the other hand the existence of an additional isolating integral would imply that the system is not ergodic and that the system phase point moves on an invariant two-dimensional manifold. For cases in which the motion is bounded, these two-

dimensional manifolds may be shown to be tori, and the system phase point executes conditionally periodic or completely periodic motion on these tori. Hamiltonians for which one can find an isolating integral in addition to the energy integral are said to be integrable.

While theoretical results as to the existence of a third integral are inconclusive, there are some empirical data which suggest the existence of an additional isolating integral. Astronomical studies of the dispersion of stellar velocities near the sun indicate the existence of a well-behaved third integral. Numerical studies of certain systems (Henon and Heiles 1964, Barbanis 1966, Perek 1966) show that for relatively low energies the third integral appears to be isolating within the limits of computational accuracy. For higher energies, i.e., larger amplitudes of motion, the integral seems to become nonisolating in some regions of phase space. This is a most surprising result since it indicates a heretofore unsuspected amplitude instability in conservative nonlinear oscillator systems. This seems to indicate that rather simple nonlinearities may provide an adequate model for physical processes provided that the amplitude of the motion is sufficiently large. The central problem of this thesis is to describe the nature and cause of this amplitude instability. In Chapter II we review the results of some recent theoretical work on conservative systems with two degrees of freedom as well as some recent numerical results which indicate the existence of a third isolating integral for small energies. Birkhoff's method for calculating this integral is discussed in Chapter III, and an essential modification of the process is made so that it will be applicable to Hamiltonian systems whose frequencies in the linear approximation are equal. This process is applied to a specific

problem in Chapter IV, and a power series expansion for an isolating integral is calculated through eighth order in the positions and momenta. This truncated integral is then used to predict the motion of a specific system, and the results are compared with those obtained by numerical integration of the equations of motion. In Chapter V the effects of various perturbation terms, called pure resonance terms, for a model Hamiltonian are examined by determining the effect of each term acting alone. These systems are in principle integrable, and it is possible to determine accurately the regions where the influence of each type of perturbation is greatest. Finally in Chapter VI we study two model Hamiltonians each having two distinct resonance terms in the perturbation acting simultaneously. All evidence indicates that these multiply-resonant systems are nonintegrable and demonstrate amplitude instability. Moreover, we are able to predict the energy and region of appearance of the amplitude instability which occurs when two or more resonant terms appear in the perturbation by treating the resonance terms as if they acted independently and by using the results of Chapter V. The accuracy of the method is established by comparing prediction with numerical integration of the equations of motion. Finally, the relevance of amplitude instability due to multiple resonance to the Henon-Heiles system is discussed in Chapter VII along with a summary of the results obtained in this study.

CHAPTER II

CONSERVATIVE DYNAMICAL SYSTEMS WITH TWO DEGREES OF FREEDOM

The most thoroughly understood conservative system (other than a free particle) with two degrees of freedom is a system of two masses moving under the influence of linear restoring and coupling forces. For physical systems, the Hamiltonian is generally a positive definite quadratic form. Hence it is possible by means of an orthogonal transformation to bring the Hamiltonian to the form (Corben and Sthele 1950)

$$H_2 = \frac{1}{2}\left[p_1^2 + p_2^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2\right], \qquad \text{(II-1)}$$

where x_1 and x_2 are the normal mode coordinates, $p_1 = \dot{x}_1$, $p_2 = \dot{x}_2$, ω_1 and ω_2 are the normal mode frequencies, and the subscript on H_2 denotes that H_2 is a quadratic form in the variables x_1, p_1, x_2, and p_2. Hamiltonian (II-1) represents a system of two independent oscillators moving with frequencies ω_1 and ω_2. In practice the forces acting on the two masses are only approximately linear, and one must add appropriate nonlinear terms to Hamiltonian (II-1). It will be assumed for the present that these nonlinear terms are functions of the position alone. With the addition of nonlinear terms, Hamiltonian (II-1) becomes

$$H = H_2 + V(x_1, x_2). \qquad \text{(II-2)}$$

In passing let us note that this Hamiltonian may be viewed as representing the motion of a single mass point in a two dimensional force field with

the origin as an equilibrium point. In some problems, the perturbation V may be sufficiently large that unbounded motion can occur. In the following we shall assume that the energy is always low enough to avoid dissociation.

For convenience in later work, we now introduce the action variables, J_1 and J_2, and the corresponding angle variables, φ_1 and φ_2, defined implicitly by the canonical transformation

$$x_i = (2J_i/\omega_i)^{1/2} \cos \varphi_i \ , \qquad \text{(II-3a)}$$

$$p_i = -(2\omega_i J_i)^{1/2} \sin \varphi_i, \ i = 1,2 \ . \qquad \text{(II-3b)}$$

Hamiltonian (II-2), expressed in these coordinates, is

$$H = \omega_1 J_1 + \omega_2 J_2 + V(J_1, \varphi_1, J_2, \varphi_2) \ . \qquad \text{(II-4)}$$

In most work, the terms depending upon the action variables alone, including those in V, are lumped together and denoted by $H_o(J_1, J_2)$. The remaining terms are then designated by V as before. Thus Hamiltonian (II-4) may be rewritten

$$H = H_o(J_1, J_2) + V(J_1, \varphi_1, J_2, \varphi_2) \ . \qquad \text{(II-5)}$$

Hamiltonian (II-5) is the most general Hamiltonian we shall consider in this thesis. Our central problem shall be the investigation of the extent to which Hamiltonian (II-5) may be considered to be integrable.

Integrable Systems

In most cases the Hamiltonian, expressed in action-angle variables, is angle-dependent and not obviously integrable. However, there is an important special case in which the Hamiltonian is immediately integrable, i.e. when

$$H = H_o(J_1, J_2) \ . \tag{II-6}$$

For Hamiltonian (II-6), the equations of motion may be written

$$\dot{J}_1 = - \frac{\partial H_o}{\partial \varphi_1} = 0 \ , \tag{II-7a}$$

$$\dot{J}_2 = - \frac{\partial H_o}{\partial \varphi_2} = 0 \ , \tag{II-7b}$$

and

$$\dot{\varphi}_1 = \frac{\partial H_o}{\partial J_1} = \Omega_1(J_1, J_2) \ . \tag{II-8a}$$

$$\dot{\varphi}_2 = \frac{\partial H_o}{\partial J_2} = \Omega_2(J_1, J_2) \ . \tag{II-8b}$$

These equations are easily integrated to yield

$$J_1 = \text{Constant}, \ J_2 = \text{Constant} \ , \tag{II-9a}$$

$$\varphi_1 = \Omega_1 t + \varphi_{10} \ , \tag{II-9b}$$

$$\varphi_2 = \Omega_2 t + \varphi_{20} \ , \tag{II-9c}$$

where Ω_1 and Ω_2 are constant frequencies which depend on the initial conditions through J_1 and J_2, and where φ_{10} and φ_{20} are the initial phase angles.

Geometrically one may view Solutions (II-9) as describing the motion of a representative point in a four-dimensional phase space which is the direct product of a two-dimensional torus and a domain of a two-dimensional Euclidean space (Arnol'd 1963). φ_1 and φ_2 are coordinates on the torus, and J_1 and J_2 are the Euclidean coordinates. Somewhat picturesquely, one may think of J_1 and J_2 as specifying the radii of the torus. The character of the motion on a torus is determined by the arithmetic nature of the frequencies Ω_1 and Ω_2. If the two frequencies are incommensurate, i.e., the equation

$$m\Omega_1 + n\Omega_2 = 0 \qquad \text{(II-10)}$$

has no integer solution for m and n other than $m = n = 0$, then the system trajectory covers the torus densely and the motion is conditionally periodic. On the other hand, if the frequencies are commensurate, i.e., Equation (II-10) has a nontrivial integer solution for m and n, then the system trajectories are closed curves and the motion is periodic.

Finally let us observe that Hamiltonian (II-1) is so obviously integrable because one can easily find an isolating third integral (either J_1 or J_2) in addition to H_2 itself. In less obvious cases, if one can by any technique find a third integral, then a theorem due to Whittaker (Whittaker 1961) guarantees that the associated Hamiltonian may be canonically transformed into Hamiltonian (II-6). We thus have the highly significant result that if Hamiltonian (II-5) is integrable, even though not

obviously so, then it may be transformed to Hamiltonian (II-6) making it obviously integrable. As a consequence, if our rather general Hamiltonian (II-5) is, in fact, integrable, we should be able to find a canonical transformation which makes integrability obvious. The next section is devoted to outlining the extent to which one may find such a canonical transformation.

Theory of Perturbations

Hamiltonian systems which are obviously integrable such as Hamiltonian (II-6) are rare exceptions; most of the ones encountered in practice have the form of Hamiltonian (II-5). The classical strategy in such cases is to treat the interaction potential V in Equation (II-5) as a perturbation and to seek a hopefully convergent sequence of canonical transformations which brings Hamiltonian (II-5) closer and closer to Hamiltonian (II-6). If this process can be realized, then one has that either J_1 or J_2 in the transformed coordinates is a third integral; one may then transform J_1, say, back to the original coordinates obtaining an isolating integral for the original Hamiltonian. Formally at least, there are a number of methods for finding the sequence of transformations which reduces the general Hamiltonian (II-5) to the obviously integrable Hamiltonian (II-6), and we shall discuss the one due to Birkhoff in detail in the next chapter. For the moment we confine ourselves to the remark that the convergence of these transformations has long been in doubt even though engineers and physicists have used them with success in numerous problems. In particular, Poincare demonstrated by example that the series of transformations does not in general converge. Moreover, Siegel (Siegel 1954) has proved

the Birkhoff method diverges in general. Thus until recently the physical scientist has had to subsist on the hope that the transformations were asymptotic in some not very well defined sense.

Fortunately recent work of Kolmogorov (Arnol'd 1963) has clarified the situation to a very great extent. Consider Hamiltonian (II-5) for the case in which V is a small perturbation. For the unperturbed system with $V = 0$, the system motion can be represented as motion on tori with various frequencies Ω_1 and Ω_2. Moreover, since the frequencies Ω_1 and Ω_2 are continuous functions of J_1 and J_2, in every neighborhood of a torus supporting conditionally periodic motion, there is a torus bearing motion which is strictly periodic with closed trajectories. Kolmogorov is able to show, under the assumption of sufficiently small perturbations, that most of the tori with conditionally periodic motion are only slightly deformed by the perturbation. However, between these "preserved" tori there are regions of instability in which the strictly periodic tori having commensurate frequencies are destroyed or grossly deformed. For a sufficiently small perturbation V, Kolmogorov is further able to show that even though the preserved tori form a nowhere dense set, almost all initial conditions (in the sense of measure theory), lie on "preserved" tori. In short, the regions of instability containing the remanents of destroyed tori have only a very small measure relative to the measure of the space. Finally, Moser (Moser 1962) has shown that the classical perturbation theories actually converge for those initial conditions which lie on the preserved tori.

Kolmogorov's theory then leads to the following simple, even though pathological, situation. Strictly speaking, a simple third isolating integral never exists and Hamiltonian (II-5) is not obviously integrable.

However, if one is content with describing the motion only on the preserved tori which span almost all the space of allowed initial conditions, then one can, as we shall show by example, obtain a relatively simple third isolating integral and Hamiltonian (II-5) is loosely speaking integrable. Finally, Kolmogorov's theory leads to the unequivical conclusion that, for small-amplitude motion, nonlinear oscillator systems are not ergodic. Thus Kolmogorov's theorem represents the greatest advance in this subject since Poincare. Despite this fact, Kolmogorov's theory says nothing about the allowed motion when the perturbation is not sufficiently small. For this we must at present turn to computer calculations which we discuss in the next section.

Some Numerical Results

In view of the theoretical uncertainly arising with large perturbations V, it is instructive to test by numerical integration of the equations of motion the existence or nonexistence of the third integral. Because of limited accuracy and relatively short integration times, computer calculations can only suggest an answer; nonetheless such calculations can be highly illuminating as we shall see. The following discussion outlines the theory needed in the computer calculations and presents the results of a computer calculation for a typical example.

Since the energy integral exists, the system phase point must move on a three-dimensional hypersurface given by

$$H(x_1, p_1, x_2, p_2) = E \qquad \text{(II-11)}$$

where E is some constant. As a consequence, we may completely specify

the system motion by giving a curve in the three dimensional space (x_1, x_2, p_2), since knowing x_1, x_2, and p_2 for some point on this curve determines P_1 through Equation (II-11). Of course if Equation (II-11) is multiple valued when solved for p_1, we must specify the desired branch.

Now if an isolating third integral exists, then the system trajectory in the (x_1, x_2, p_2) space is further constrained by

$$I(x_1, p_1, x_2, p_2) = I_o \ , \tag{II-12}$$

where I_o is a constant. In fact we may solve Equation (II-11) for p_1, choosing the desired branch, and then put the resulting expression for p_1 into Equation (II-12) obtaining

$$I(x_1, x_2, p_2) = I_o \ . \tag{II-13}$$

Equation (II-13) shows that the system trajectory in the (x_1, x_2, p_2) space moves on a two dimensional surface. (In short, as determined previously the system trajectory lies on a torus.) This two dimensional surface or torus intersects the $x_2 - p_2$ plane, defined by $x_1 = 0$, through the curve specified by

$$I(0, x_2, p_2) = I_o \tag{II-14}$$

where Equation (II-14) is obtained from Equation (II-13) through the substitution $x_1 = 0$. Thus if an isolating third integral exists, the intersections of a given trajectory in (x_1, x_2, p_2) space with the $x_2 - p_2$ plane lie on a curve called a level curve. Varying I_o for fixed energy E yields a one parameter family of level curves.

On the other hand, if no isolating third integral exists, then each (x_1, x_2, p_2) system trajectory would densely fill the allowed three dimensional energy volume. Moreover, the system trajectory in (x_1, x_2, p_2) space would densely penetrate some area in the $x_2 - p_2$ plane. Briefly then, in regions of phase space where motion is on tori and an isolating third integral exists, one anticipates finding smooth level curves in the $x_2 - p_2$ plane. In regions where the tori have broken down and an isolating third integral does not exist, one expects random scatter of the intersection points in the $x_2 - p_2$ plane.

As a consequence, we have a straightforward scheme for determining the presence or absence of an isolating third integral. One integrates the equations of motion for a given set of initial conditions. Then each time $x_1 = 0$, one plots the corresponding value of x_2 and p_2 in the $x_2 - p_2$ plane. If the points lie on a curve, then an isolating integral is presumed to exist. If the points are randomly scattered, then one assumes that no isolating integral exists. This process is repeated for a representative sampling of initial conditions for a given energy and then for various energies. Thereby one determines the character of the motion throughout phase space. While the method has obvious shortcomings, we may illustrate its usefulness by an example.

Although several Hamiltonians have been successfully studied using the level curve technique, in this section we confine our attention to one illustrative example due to Henon and Heiles (Henon and Heiles 1964). They considered the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + (V/2) , \qquad \text{(II-15)}$$

where

$$V = x_1^2 + x_2^2 + 2x_1^2 x_2 - (2x_2^3/3) . \qquad \text{(II-16)}$$

The potential energy function of Equation (II-16) has equipotential lines which are a series of ovals, centered on the origin of the $x_1 - x_2$ plane, bounded by an equilateral triangle. Inside the triangle motion is bounded; while outside the triangle, motion becomes unbounded. The transition occurs at $E = 1/6$, and Henon and Heiles confined their attention to energies less than or equal to one-sixth. Solving Equation (II-15) for p_1 yields two branches, and Henon and Heiles chose to plot level curves for the positive square root branch, i.e., they plotted values of $x_2 - p_2$ on each trajectory corresponding to $x_1 = 0$ and $p_1 \geq 0$.

For energies $E \leq .10$, Henon and Heiles found that the computed points in the $x_2 - p_2$ plane all lay on smooth curves. We have repeated their calculations for several energies on a UNIVAC 1108 Computer using a classical fourth order Runge-Kutta technique (Hildebrandt 1948) with integration step size $H = 0.05$. After each integration step, x_1 was checked for a sign change indicating the passage of a zero in x_1. When this occurred, the integration step size was reduced and the integration was repeated until the absolute value of x_1 was less than some preselected value, usually 10^{-4}. If p_1 was found to be greater than or equal to zero, the values of x_2 and p_2 were then recorded. As a check on the integration, the energy was calculated at each recorded point and was observed to be conserved within six figure accuracy. In addition, various orbits were integrated for about one thousand seconds of differential equation time

and then the signs of the momenta p_1 and p_2 were reversed and integration continued back to the initial point. These reversed trajectory calculations maintained an accuracy to about four figures which is greater than the graphical accuracy used in plotting the curves.

The level curves for E = 1/12 are shown in Figure 1. There are seven level curves in Figure 1 which consist of a single point. They represent periodic solutions. The points lying on the x_2 and p_2 axes, four in all, are stable periodic solutions. The three intersection points on the peculiar self-intersecting curve are unstable periodic solutions. The self-intersecting curve itself separates the four sets of curves surrounding the four stable periodic solutions and for that reason is called a separatrix. Motion on the separatrix curve is unique in that one or another of the unstable periodic solutions is approached asymptotically. In essence motion on the separatrix is analogous to the motion of a pencil started such that the pencil will asymptotically approach its unstable vertical equilibrium position. The entire set of level curves is bounded by the curve for which both x_1 and p_1 are zero. This bounding curve happens to be also a periodic solution for Hamiltonian (II-15) as may be verified almost by inspection.

In Figure 2 a similar set of curves is shown for the energy E = .1063. Some new features are visible such as the somewhat irregular cluster of eight curves around the central invariant point on the positive p_2 axis. This cluster is a single level curve and successive points on this level curve hop sequentially to each of the group of eight closed curves. Such systems of curves are called "chains of islands." The central invariant points of these islands represent a single periodic solution. A set of

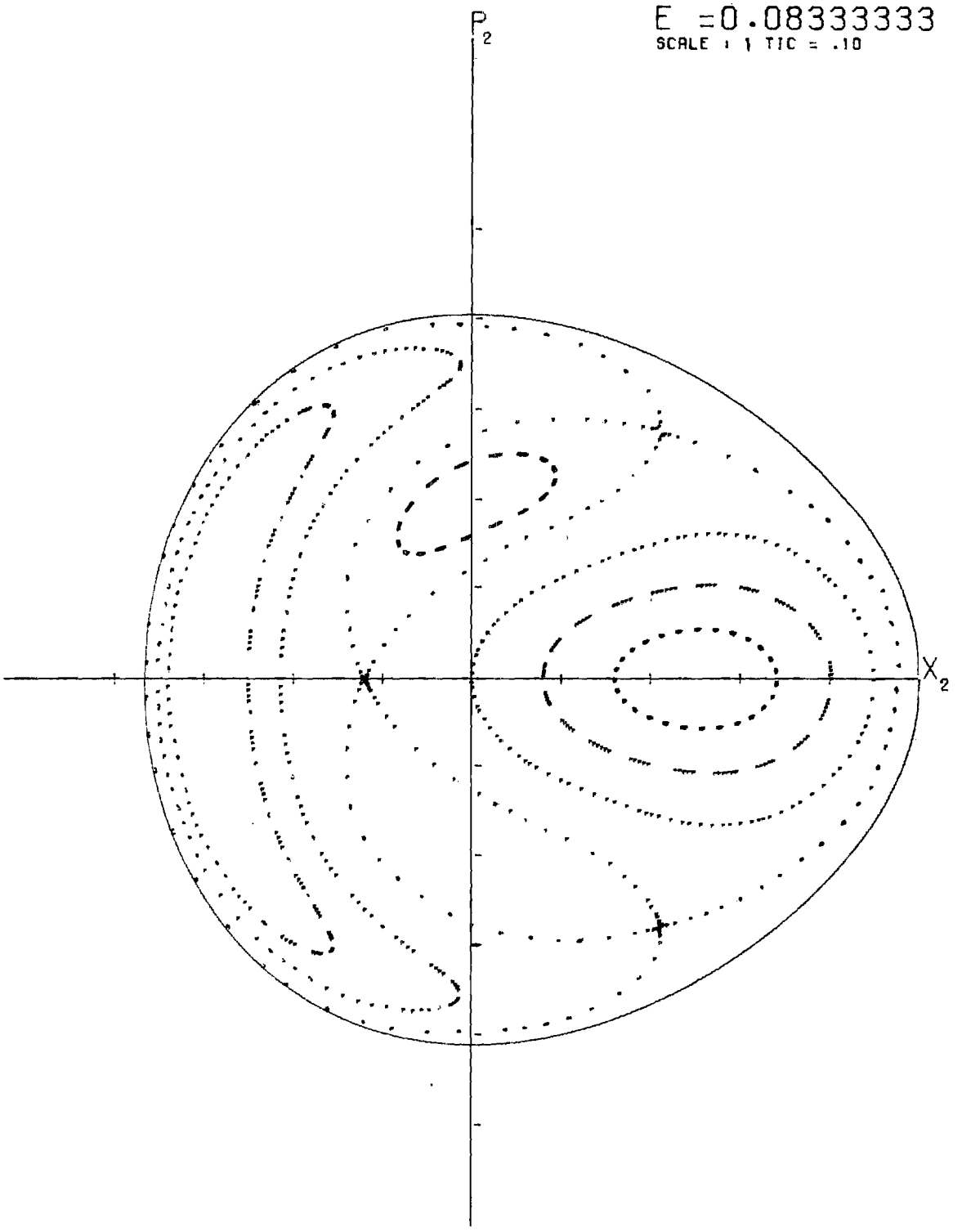

Figure 1. Level Curves for the Henon and Heiles System.

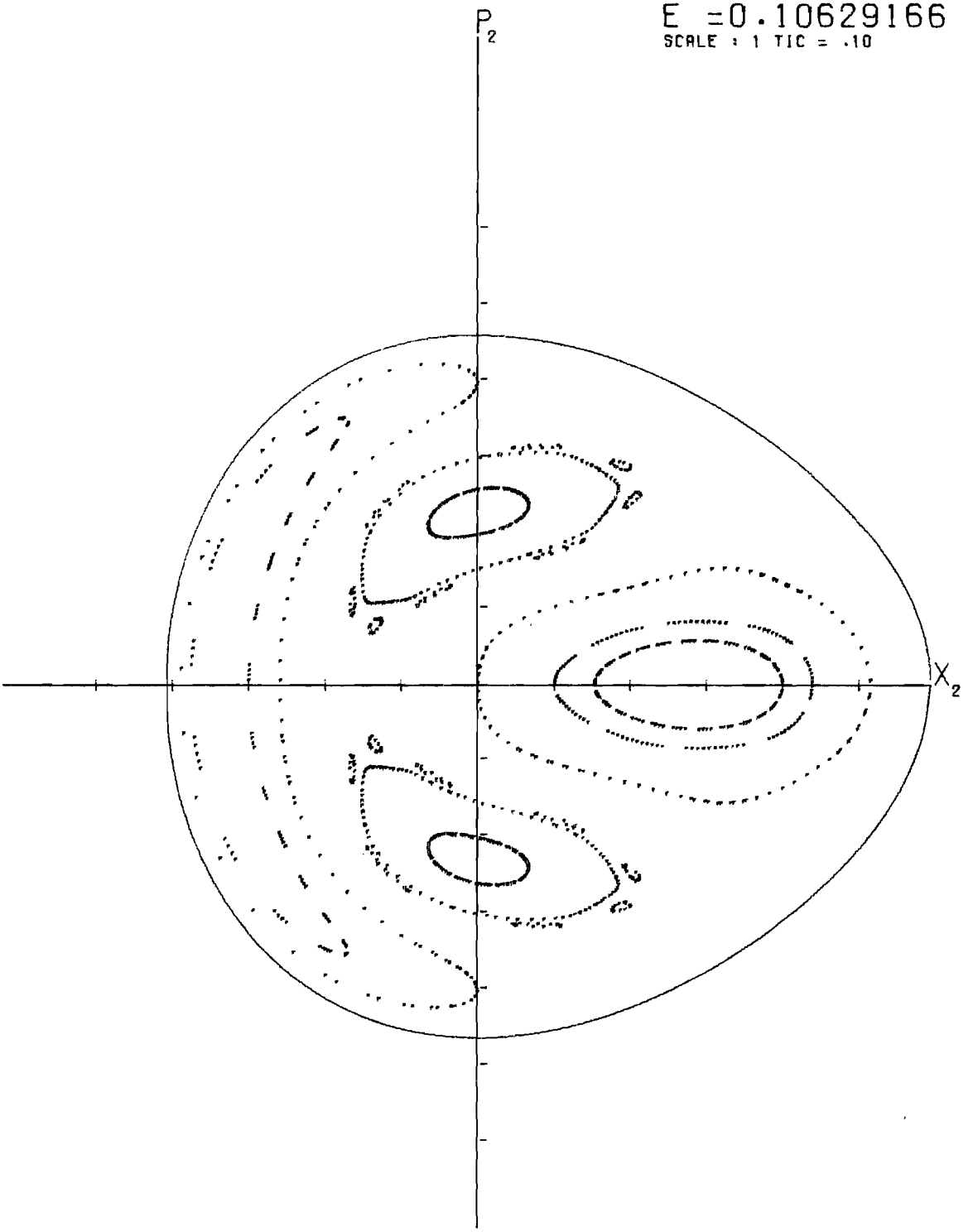

Figure 2. Level Curves for the Henon and Heiles System.

points for an orbit with initial conditions close to the orbit which produced the chain of eight islands is shown in Figure 3. Here it is difficult to tell whether or not the points lie on a simple curve. Because there is an unstable periodic solution associated with the separatrix for this island chain, it may be necessary to use more accurate integration techniques here. Possibly if this were done, a smooth curve could be obtained. In addition a set of points resulting from an orbit close to the original separatrix is plotted in Figure 3. Thus there are two sets of irregular points in this diagram.

At an energy of $E = 1/8$ the picture changes even more radically. The results for this energy are shown in Figure 4. The separatrix of Figure 1 has disappeared, and there are several new chains of islands, such as the chain of five about the central invariant point on the positive x axis. Other island chains have been found but are not shown in this figure. Some of the level curves remain. However there are certain initial conditions which give points that do not seem to lie on any curve. One such set of points is shown in Figure 4. In fact, there is a sizable "breakdown" area in the plane which seems to contain relatively few smooth level curves. As the energy increases, this breakdown area increases until at the dissociation energy, $E = 1/6$, shown in Figure 5, the area containing closed curves is very small indeed (Gustavson 1966). The set of points lying on no curve in Figure 5 all result from one orbit.

These numerical experiments suggest that for lower energies there is an isolating third integral which exists everywhere. At higher energies, this integral appears to change its character quite abruptly at about $E = 0.11$ from isolating in some regions to nonisolating in others. The

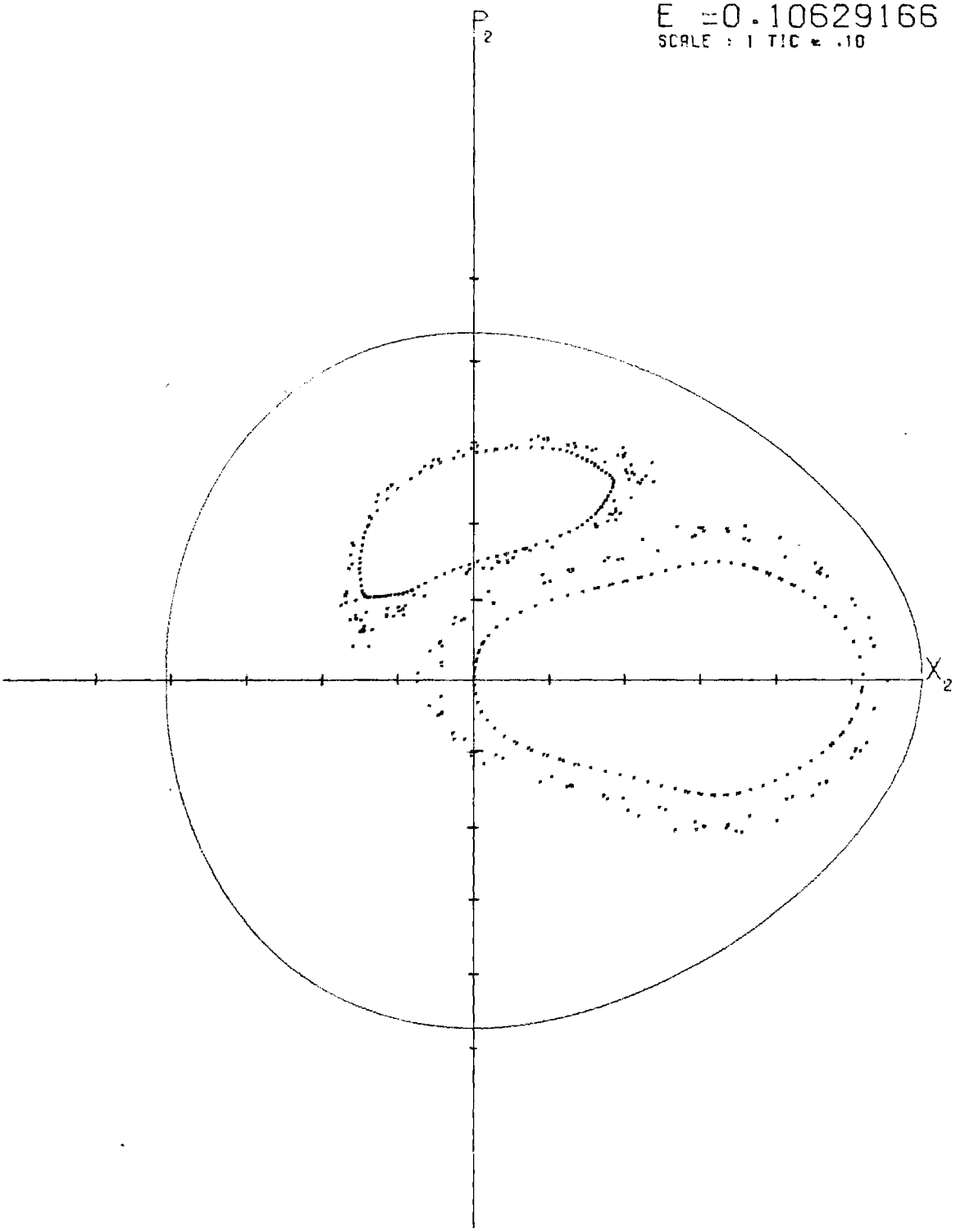

Figure 3. Level Curves for the Henon and Heiles System.

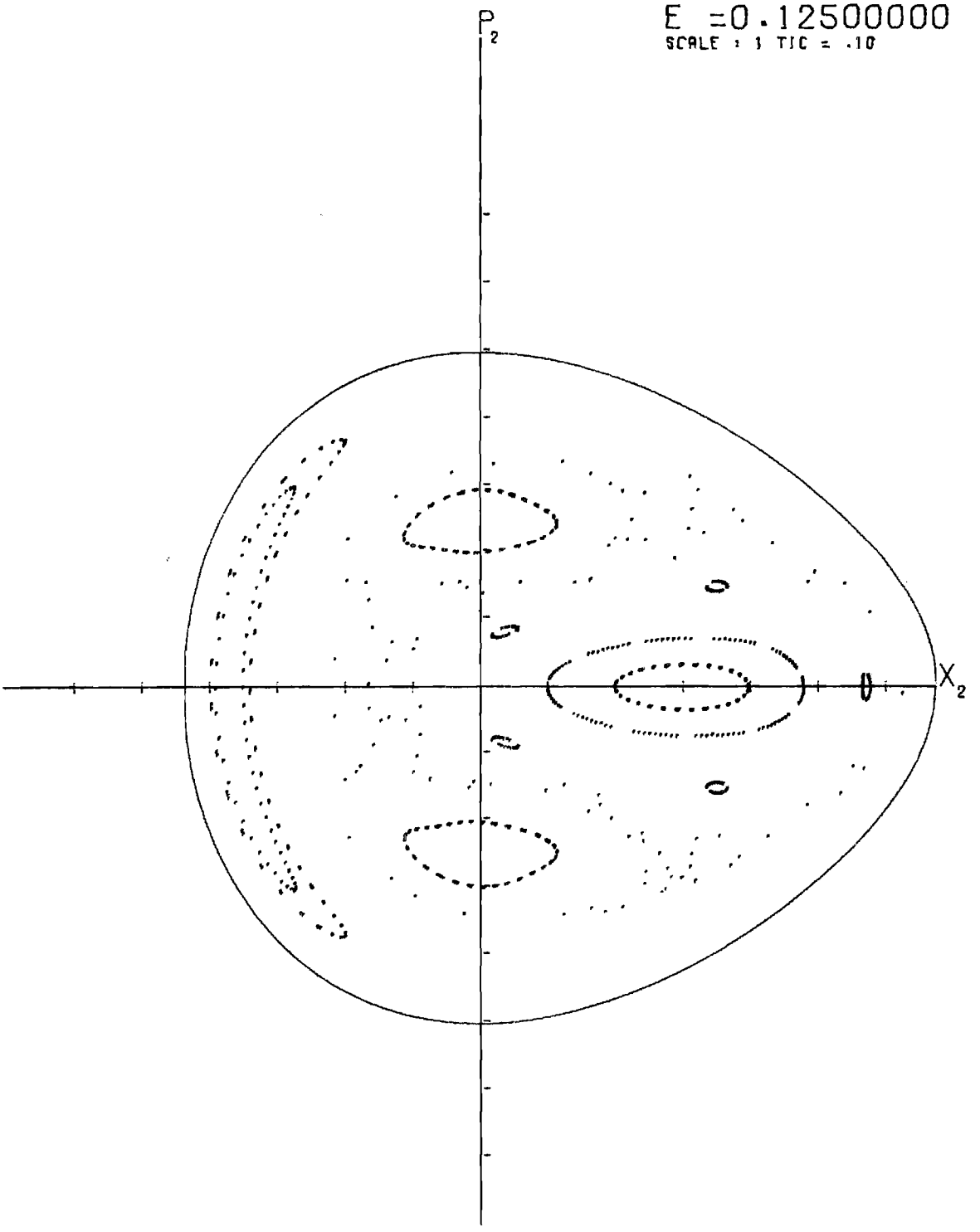

Figure 4. Level Curves for the Henon and Heiles System.

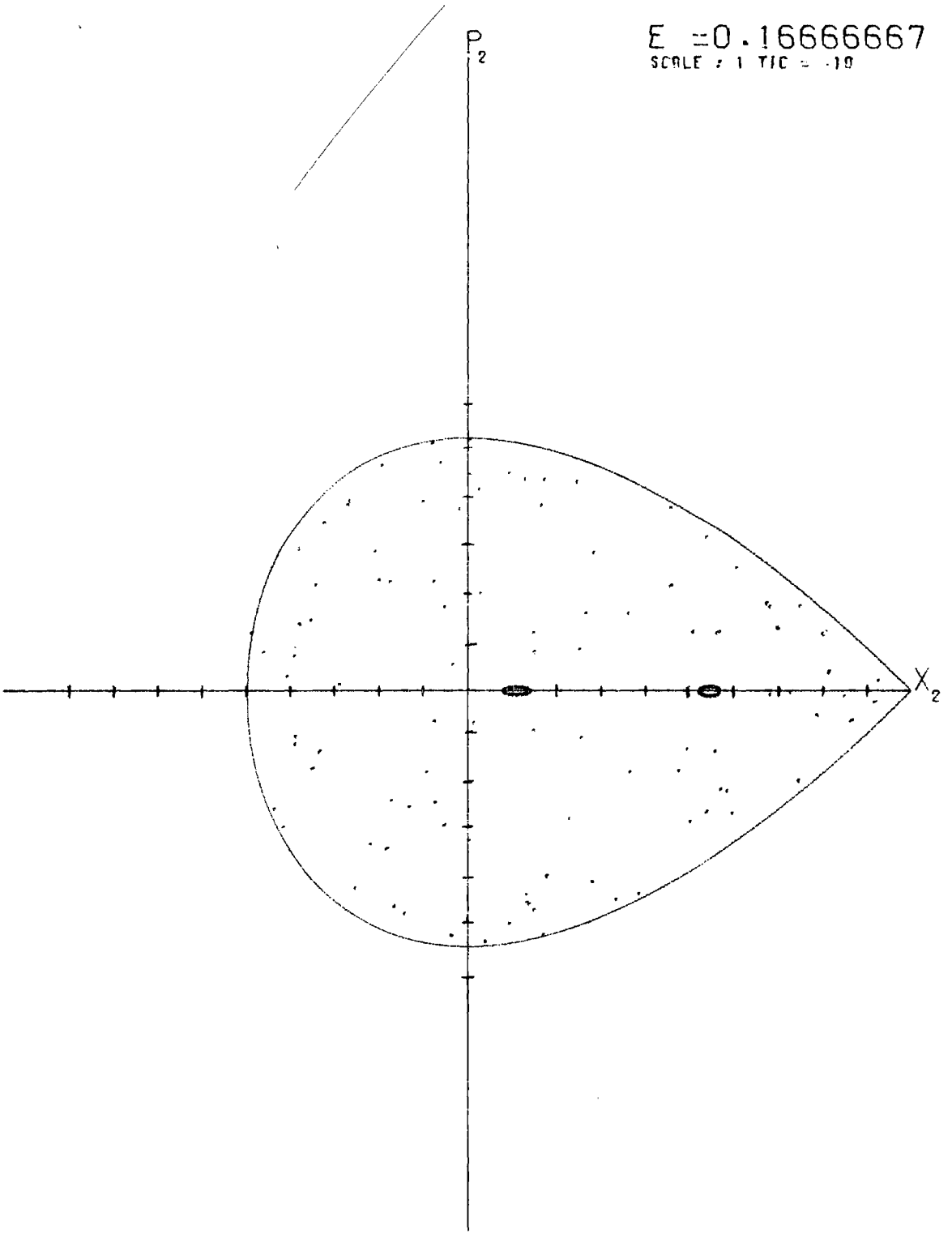

Figure 5. Level Curves for the Henon and Heiles System.

region where the integral is nonisolating increases with increasing energy. The behavior is typical of other systems that have been studied; although certain potentials have been shown not to lead to a breakdown in the third integral, which continues to exist even in the dissociation region (Henon and Heiles 1966).

The problems to be attacked in this thesis can now be clearly stated. First, we must develop a scheme for calculating the isolating third integral for those preserved tori which continue to exist. In passing we note that for small amplitude motion, the preserved tori as illustrated in the Henon-Heiles calculations, are so close together that the breakdown cannot be observed on a computer. In any event, having analytically calculated an isolating third integral, we must compare the level curves predicted by this integral with the level curves obtained by direct integration of the equations of motion. Finally, we must attempt to provide an analytic explanation for this sudden onset of the disappearance of the third integral from increasing regions of phase space. We then may check the analytic predictions of breakdown with direct integration of the equations of motion. In the following chapter, we begin by presenting a procedure for analytically calculating an isolating third integral.

CHAPTER III

THE BIRKHOFF DIAGONALIZATION PROCESS

The numerical results presented in Chapter II suggest strongly the existence of an isolating third integral for small amplitude motion. Logically the next step is to calculate an analytic expression for this integral and to see if the results enable one to predict the curves obtained by Henon and Heiles. Of course in practice one develops a series expansion for this integral which must be truncated at some finite degree. Still, for small energies and displacements, it should be possible to use the truncated integral and the Hamiltonian to calculate a fairly good approximation to the level curves in the x_2 - p_2 plane.

The easiest method for obtaining a power series expansion of the third integral (from the standpoint of possible computerization) is a method due to Birkhoff (Birkhoff 1927). This expansion comes about in a natural way when reducing the Hamiltonian to an obviously integrable form (normal form). We start with the Hamiltonian of Equation (II-2) in Chapter II and we assume that $V(x_1,x_2)$ can be expanded in a power series about the origin starting with terms of degree three or higher. We also assume that ω_1 and ω_2, the uncoupled oscillator frequencies, are incommensurate. To perform the reduction to normal form, complex coordinates, Q_j and P_j are introduced via the canonical transformation

$$x_j = \frac{Q_j + iP_j}{\sqrt{2\omega_j}}, \qquad \text{(III-1a)}$$

$$p_j = \sqrt{\frac{\omega_j}{2}}\ (P_j + iQ_j); \quad j = 1,2\ . \qquad \text{(III-1b)}$$

where $i^2 = -1$. Hamiltonian (II-2) becomes

$$H = i\omega_1 Q_1 P_1 + i\omega_2 Q_2 P_2 + V(Q_1, P_1, Q_2, P_2). \qquad \text{(III-2)}$$

The perturbing term in the potential is expanded in a power series

$$V = V_3 + V_4 + \ldots + V_n + \ldots, \qquad \text{(III-3)}$$

where V_n represents a homogeneous polynomial of degree n in the variables Q_1, P_1, Q_2, and P_2. The reduction procedure will consist of the following steps: A canonical transformation to new variables is calculated such that in the new variables the third order terms in V vanish. Using the new Hamiltonian, another transformation is determined so that as many as possible of the fourth order terms in V vanish, etc. For purposes of discussion, it will be assumed that this reduction has been performed for all terms of degree less than n. The normalized terms will then be denoted by $\bar{V}_R$. Thus, we assume that

$$H = i\omega_1 Q_1 P_1 + i\omega_2 Q_2 P_2 + \bar{V}_3 + \ldots + \bar{V}_{n-1} + V_n + \ldots \quad . \qquad \text{(III-4)}$$

Following Goldstein (Goldstein 1951) we seek the desired transformation to new coordinates $\bar{Q}_1$, $\bar{P}_1$, $\bar{Q}_2$, and $\bar{P}_2$, for which V_n is reduced to normal form by introducing a generating function

$$K = Q_1\bar{P}_1 + Q_2\bar{P}_2 + K_n(Q_1, \bar{P}_1, Q_2, \bar{P}_2)\ , \qquad \text{(III-5)}$$

where K_n is homogeneous of degree n with undetermined coefficients. The

desired transformation is obtained from the equations

$$\bar{Q}_j = \frac{\partial K}{\partial \bar{P}_j} ; \qquad \text{(III-6a)}$$

$$P_j = \frac{\partial K}{\partial Q_j} ; \quad j = 1, 2 . \qquad \text{(III-6b)}$$

Using Equation (III-5), we may write Equation (III-6) as

$$\bar{Q}_j = Q_j + \frac{\partial K_n}{\partial \bar{P}_j} , \qquad \text{(III-7a)}$$

$$P_j = \bar{P}_j + \frac{\partial K_n}{\partial Q_j} . \qquad \text{(III-7b)}$$

If the displacements and velocities are small, Equation (III-7) is almost the identity transformation, and we solve these equations by iteration. The first approximation is $Q = \bar{Q}_j$, $P_j = \bar{P}_j$. The next order of approximation is

$$Q_j = \bar{Q}_j - \frac{\partial K_n}{\partial \bar{P}_j} (\bar{Q}_1, \bar{P}_1, \bar{Q}_2, \bar{P}_2) , \qquad \text{(III-8a)}$$

$$P_j = \bar{P}_j + \frac{\partial K_n}{\partial \bar{Q}_j} (\bar{Q}_1, \bar{P}_1, \bar{Q}_2, \bar{P}_2); \quad j = 1,2 . \qquad \text{(III-8b)}$$

In this notation the symbol $\frac{\partial K_n}{\partial \bar{Q}_1} (\bar{Q}_1, \bar{P}_1, \bar{Q}_2, \bar{P}_2)$, for example, means that the partial derivative of $K(Q_1, \bar{P}_1, Q_2, \bar{P}_2)$ is taken with respect to the first argument and then the substitutions $Q_1 = \bar{Q}_1$ and $Q_2 = \bar{Q}_2$ are made in the resulting function. If one continues iterating, it is possible to

obtain a power series representation for the transformation defined by Equation (III-6). Using Transformation (III-8) which is correct through order n-1, one obtains

$$H = i\omega_1 Q_1 P_1 + i\omega_2 Q_2 P_2 + \bar{V}_3 + \bar{V}_4 + .. + \bar{V}_{n-1} + i\left(\omega_1 \bar{Q}_1 \frac{\partial K_n}{\partial \bar{Q}_1} - \omega_1 \bar{P}_1 \frac{\partial K_n}{\partial \bar{P}_1}\right.$$

$$\left. + \omega_2 \bar{Q}_2 \frac{\partial K_n}{\partial \bar{Q}_2} - \omega_2 \bar{P}_2 \frac{\partial K_n}{\partial \bar{P}_2}\right) + V_n(\bar{Q}_1, \bar{P}_1, \bar{Q}_2, \bar{P}_2) + \ldots . \qquad \text{(III-9)}$$

retaining terms only through nth order in the variables $\bar{Q}_1, \bar{P}_1, \bar{Q}_2$, and $\bar{P}_2$. The function K_n has not yet been specified; it will be determined so that as many as possible of the nth order terms vanish. Let α be the coefficient of $\bar{Q}_1^s \bar{P}_1^t \bar{Q}_2^u \bar{P}_2^v$ in V_n with $s+t+u+v = n$, and let γ be the corresponding coefficient in K_n. Then we may verify by direct substitution that this term is eliminated from V_n if we require that

$$\gamma = \frac{i\alpha}{\omega_1(s-t) + \omega_2(u-v)} . \qquad \text{(III-10)}$$

Since by hypothesis ω_1 and ω_2 are incommensurate, it is impossible for the denominator in Equation (III-10) to vanish unless $s = t$ and $u = v$. Since $s = t$ and $u = v$ is not possible for odd order terms, a zero denominator can never occur; hence all terms V_n for odd n can be completely eliminated. For even order terms, the denominator in Equation (III-10) vanishes for $s = t$ and $u = v$ which means that terms in the Hamiltonian of the form $\alpha \bar{Q}_1^s \bar{P}_1^s \bar{Q}_2^u \bar{P}_2^u$ cannot be eliminated. Hence an even order V_n can ultimately be reduced to products of $(\bar{Q}_1 \bar{P}_1)$ and $(\bar{Q}_2 \bar{P}_2)$. Specifically, V_n may be written as

$$V_n = \sum_{\ell=o}^{n-j-k} \sum_{k=o}^{n-j} \sum_{j=o}^{n} \phi_{j,k,\ell,n-j-k-\ell} Q_1^{\,j} P_1^{\,k} Q_2^{\,\ell} P_2^{\,n-j-k-\ell} \qquad \text{(III-11)}$$

Using Equation (III-10), we find that the required generating function which normalizes V_n is

$$K = Q_1\bar{P}_1 + Q_2\bar{P}_2 + \sum_{\ell=o}^{n-j-k} \sum_{k=o}^{n-j} \sum_{j=o}^{n} {}' \frac{i\alpha_{j,k,\ell,n-j-k-\ell} Q_1^{\,j} \bar{P}_1^{\,k} Q_2^{\,\ell} \bar{P}_2^{\,n-j-k-\ell}}{\omega_1(j-k) + \omega_2(2\ell+j+k-n)} \qquad \text{(III-12)}$$

A prime on a set of summation symbols will be used to indicate that values of the indices which would give zero in the denominator are to be deleted in the sum. Using the canonical transformation generated by Equation (III-12) in Hamiltonian (III-4) gives a new Hamiltonian normalized through degree n. The process is repeated until the desired degree of normalization is attained.

A Hamiltonian that has been transformed in this fashion is a function of the products $(\bar{Q}_1\bar{P}_1)$ and $(\bar{Q}_2\bar{P}_2)$ alone and is said to be in Birkhoff's normal form. Action-angle variables may now be introduced using Equations (III-1) and (II-3) or equivalently one may use

$$\bar{Q}_j = \sqrt{J_j}\, e^{i\varphi_j} \qquad \text{(III-13a)}$$

$$\bar{P}_j = \frac{\sqrt{J_j}}{i}\, e^{-i\varphi_j} \qquad \text{(III-13b)}$$

Since $\bar{Q}_j\bar{P}_j = J_j/i$, we have that Q_jP_j (dropping the bars) is proportional to J_j. Thus any Hamiltonian in normal form is a function of the action

alone and hence may be easily integrated as was shown in Chapter II. It is interesting to note at this point that the Birkhoff normalization procedure leads to the appearance of small denominators. Even though ω_1 and ω_2 are incommensurate, for very large values of s,t,u,v, the denominator $(s-t)\omega_1 + (u-v)\omega_2$ can be arbitrarily close to zero. These small denominators obviously endanger the convergence of the normalization process; in particular Siegel (Siegel 1954) has shown that convergence almost never occurs.

We now turn to the case which was not considered by Birkhoff, namely the case for which the frequencies are commensurate and there exist nonzero integers m and n such that

$$m\omega_1 + n\omega_2 = 0 \ . \qquad \text{(III-14)}$$

This means that in Equation (III-10) or (III-12) values of s,t,u, and v other than s = t and u = v will lead to zero denominators. It will thus be impossible to reduce the Hamiltonian to a function of the action alone because the terms $\bar{Q}_1^s \bar{P}_1^t \bar{Q}_2^u \bar{P}_2^v$ for which $\omega_1(s-t) + \omega_2(u-v) = 0$, called resonant terms, cannot be eliminated. The name "resonant terms" arises because, as one may show, these terms give rise to energy sharing between the two oscillators. We shall in a later chapter investigate in detail the effect of these resonant terms on the motion of the system. For the present, we note only that it is still possible to perform a useful normalization.

For simplicity only the case for which $\omega_1 = \omega_2$ will be treated, and it will be assumed that units have been chosen so that $\omega_1 = 1$. The extension of this method to other cases is straightforward. Requiring

that $\omega_1 = \omega_2 = 1$, Hamiltonian (III-2) becomes

$$H = iQ_1P_1 + iQ_2P_2 + V_3 + V_4 + \ldots \qquad \text{(III-15)}$$

Proceeding as before, no resonant terms occur in any V_n for n odd; hence all of the odd order V_n terms may be eliminated. When we attempt to normalize V_4, however, we find that it may be reduced to

$$V_4 = a\bar{Q}_1^2\bar{P}_1^2 + b\bar{Q}_2^2\bar{P}_2^2 + c\bar{Q}_1\bar{P}_1\bar{Q}_2\bar{P}_2 + d\bar{Q}_1^2\bar{P}_2^2 + e\bar{Q}_2^2\bar{P}_1^2, \qquad \text{(III-16)}$$

with a,b,c,d, and e being constants. In terms of action-angle variables,

$$V_4 = -(aJ_1^2 + bJ_2^2 + cJ_1J_2 + d\sqrt{J_1J_2}\, e^{2i(\varphi_1-\varphi_2)} + e\sqrt{J_1J_2}\, e^{-2i(\varphi_1-\varphi_2)}) \qquad \text{(III-17)}$$

For even order terms in general, it will be impossible to eliminate terms for which $s - t + u - v = 0$. Using action-angle variables, we find in general that $Q_1^s P_1^t Q_2^u P_2^v$ is proportional to $\exp[i(s-t)\varphi_1 + i(u-v)\varphi_2]$. In particular, we have $\exp[i(s-t)(\varphi_1 - \varphi_2)]$ when $s-t = -(u-v)$. Thus in the case $\omega_1 = \omega_2 = 1$, Birkhoff normalization reduces the Hamiltonian to the form

$$H = H(J_1, J_2, \varphi_1 - \varphi_2) , \qquad \text{(III-18)}$$

While Hamiltonian (III-18) is not a function of the action alone, there nonetheless is an integral of motion in addition to the energy, namely

$$I = J_1 + J_2 . \qquad \text{(III-19)}$$

To prove this, we calculate $\dot{I}$ and use the equations of motion.

$$\dot{I} = \dot{J}_1 + \dot{J}_2 = - \frac{\partial H}{\partial \varphi_1} - \frac{\partial H}{\partial \varphi_2} = 0 . \qquad \text{(III-20)}$$

Since there are two independent integrals, the equations of motion may be integrated (Whittaker 1961). Indeed Whittaker shows that a transformation exists which carries Hamiltonian (III-18) into a Hamiltonian which is a function of the action alone. We do not attempt this last generally complicated transformation since, as shown by Equation (III-20), we may obtain a simple isolating integral without further complications. The problem of small denominators does not obviously occur in the normalization of this resonant Hamiltonian; indeed the denominators are all non-zero integers. Nonetheless convergence is still unlikely for reasons discussed later in Chapter IV.

From a practical standpoint Birkhoff normalization has a serious disadvantage. It is generally found that introducing real Cartesian coordinates into the normal form Hamiltonian in a manner analogous to Equation (III-1) does not lead to a real form. Since a real normal form in Cartesian coordinates is highly desirable, one needs a modification of the Birkhoff method leading to a normal form and to transformations which are real when expressed in real coordinates. In the following section we develop the needed modifications.

Normalization Using Real Transformations

To circumvent the aforementioned difficulties, it is necessary to modify the Birkhoff process considerably (for an alternate method published while this work was in progress see Gustavson, 1966). The desired modifi-

cation can be accomplished by performing a sequence of canonical transformations using only real coordinates in such a way that the final real Hamiltonian is in normal form when expressed in complex coordinates using Transformation (III-1). Hamiltonian (III-15) in terms of real coordinates is

$$H = \frac{p_1^2 + p_2^2}{2} + \frac{x_1^2 + x_2^2}{2} + V_3(x_1,x_2)+... \qquad \text{(III-21)}$$

In a fashion similar to the Birkhoff procedure, we shall try to find a function which generates a canonical transformation to new coordinates in which the coefficients of the terms of a given order are altered so that the Hamiltonian is in normal form through order n. We shall use complex coordinates in this process, although they are not logically necessary.

Before describing this process in detail, it is expedient to point out certain relationships that must be satisfied by the complex coordinates Q_j and P_j. The transformation equations inverse to those of Equation (III-1) imply that

$$Q_j^* = iP_j; \quad P_j^* = iQ_j \ , \qquad \text{(III-22)}$$

where complex conjugation will be indicated throughout by an asterisk. One may show using Equation (III-11) and (III-22) that as a consequence of starting with a real Hamiltonian the α's must satisfy

$$\alpha_{j,k,\ell,n-j-k-\ell} = i^n \alpha^*_{k,j,n-j-k-\ell,\ell} \ . \qquad \text{(III-23)}$$

Having made these preliminary remarks, we now assume that a series of real transformations has been made such that all terms of order less than n in Equation (III-21) are in normal form. In particular,

$$H = \frac{p_1^2 + p_2^2}{2} + \frac{x_1^2 + x_2^2}{2} + \bar{V}_3 + \bar{V}_4 + \ldots + \bar{V}_{n-1} + V_n + \ldots \quad \text{(III-24)}$$

We shall now find a real transformation which normalizes V_n. In order to do this, H given by Equation (III-24) is transformed to the equivalent Hamiltonian expressed in complex coordinates using Transformation (III-1). We then have

$$H = iQ_1P_1 + iQ_2P_2 + \bar{V}_3 + \ldots + \bar{V}_{n-1} + V_n + \ldots \quad \text{(III-25)}$$

As in the preceding section a generating function is calculated such that the resulting transformation normalizes V_n in Equation (III-25). The first order approximation to this transformation is given by Equation (III-8), and, by using the iteration scheme mentioned previously, we can express the original complex coordinates Q_j, P_j, as a power series in the new complex coordinates $\bar{Q}_j$, $\bar{P}_j$. For the present case we must in addition express the original real Cartesian coordinates x_1, p_1, x_2, and p_2 in terms of the new complex coordinates $\bar{Q}_j$, $\bar{P}_j$. Using Equations (III-8) and (III-1) we have that

$$x_1 = \frac{\bar{Q}_j + i\bar{P}_j}{2} + \frac{1}{2}\left(i\frac{\partial K_n}{\partial \bar{Q}_j} - \frac{\partial K_n}{\partial \bar{P}_j}\right) + \ldots , \quad \text{(III-26a)}$$

$$p_j = \frac{\bar{P}_j + i\bar{Q}_j}{V_2} + \frac{1}{V_2}\left(\frac{\partial K_n}{\partial \bar{Q}_j} - i\frac{\partial K_n}{\partial \bar{P}_j}\right) + \ldots . \quad \text{(III-26b)}$$

The right hand side of Equations (III-26a) and (III-26b) must be real by virtue of the fact that x_j and p_j are real. However, the four complex coordinates $\bar{Q}_j$, $\bar{P}_j$ represent a total of eight real numbers not all of which can be independent. In fact there must be a set of four relations, or reality conditions, involving these complex coordinates in such a way that x_j and p_j turn out to be real. An example of an especially simple set of reality conditions is Equation (III-22).

In order to motivate our choice of reality conditions we note that in later work it will be necessary to introduce new real Cartesian variables $\bar{x}_j$ and p_j, where

$$\bar{x}_j = \frac{\bar{Q}_j + i\bar{P}_j}{\sqrt{2}} , \qquad \text{(III-27a)}$$

$$\bar{p}_j = \frac{\bar{P}_j + i\bar{Q}_j}{\sqrt{2}} . \qquad \text{(III-27b)}$$

The transformation (III-27) is canonical and implies that the reality conditions for $\bar{Q}_j$ and $\bar{P}_j$ are

$$\bar{Q}_j^* = i\bar{P}_j , \qquad \text{(III-28a)}$$

$$\bar{P}_j^* = i\bar{Q}_j ; \quad j = 1,2 . \qquad \text{(III-28b)}$$

However one may show that these reality conditions are not necessarily consistent with the reality conditions for Equation (III-26), retaining terms of all orders. However there is consistency if only the first and second set of terms are retained in Equation (III-26). To see that this is true, we must explicitly write out the second term in Equation (III-26).

Using Equation (III-12) we find that

$$i\frac{\partial K_n}{\partial \bar{Q}_1} - \frac{\partial K_n}{\partial \bar{P}_1} = \sum\sum\sum{}' \frac{i\alpha_{j,k,\ell,n-j-k-\ell}}{\omega_1(j-k)+\omega_2(2\ell+j+k-n)}\left[ij\bar{Q}_1^{j-1}\bar{Q}_2^{\ell}\bar{P}_1^{k}P_2^{n-j-k-\ell} - k\bar{Q}_1^{j}\bar{P}_1^{k-1}\bar{Q}_2^{\ell}\bar{P}_2^{n-j-k-\ell}\right] \tag{III-29}$$

Similar expressions can be calculated for the other terms appearing in Equation (III-26). The complex conjugate of Equation (III-29) is

$$i\frac{\partial K_n}{\partial \bar{Q}_1} - \frac{\partial K_n}{\partial \bar{P}_1}^{*} = \sum\sum\sum{}' \frac{\alpha_{k,j,n-j-k-\ell,\ell}}{\omega_1(j-k)+\omega_2(2\ell+j+k-n)}\left[-(-i)^n j(\bar{Q}_1^{*})^{j-1}(\bar{P}^{*})^{k} \cdot (\bar{Q}_2^{*})^{\ell}(\bar{P}_2^{*})^{n-j-k-\ell} + i(-i)^n k(\bar{Q}_1^{*})^{j}(\bar{P}_1^{*})^{k-1}(\bar{Q}_2^{*})^{\ell}(\bar{P}_2^{*})^{n-j-k-\ell}\right], \tag{III-30}$$

where use has been made of Equation (III-23). If the reality conditions given by Equation (III-28) are used in Equation (III-30), one may show that

$$\left[i\frac{\partial K_n}{\partial \bar{Q}_1} - \frac{\partial K_n}{\partial \bar{P}_1}\right]^{*} = i\frac{\partial K_n}{\partial \bar{Q}_1} - \frac{\partial K_n}{\partial \bar{P}_1} . \tag{III-31}$$

This result is also true for the analogous expressions arising in Equation (III-26) for x_2, p_1 and p_2. Therefore, if we truncate Equation (III-26) at the first iteration and introduce new Cartesian coordinates by means of Equation (III-27), we obtain a truncated expression for a real transformation between two sets of real coordinates. We shall write this transformation as

$$x_i = \bar{x}_i + \psi_i(\bar{x}_1, \bar{p}_1, \bar{x}_2, \bar{p}_1) \ . \qquad \text{(III-32a)}$$

$$p_i = \bar{p}_i + \varphi_i(\bar{x}_1, \bar{p}_1, \bar{x}_2, \bar{p}_2) \ . \qquad \text{(III-32b)}$$

ψ_i and φ_i are homogeneous polynomials of degree (n-1) and are obtained by substituting the inverse of Equation (III-27) into Equation (III-26). This truncated transformation is not canonical, but if it is applied to Hamiltonian (III-24), the nth order term is clearly normalized.

Since it is desirable to preserve the formal structure of Hamilton's equations of motion, we modify Transformation (III-32) by adding terms of higher order in such a way that the resulting transformation is canonical. This can be done by choosing a generating function that generates in zeroth order the identity transformation and through first order yields Equation (III-32). We shall for the moment assume that a generating function with these properties exists and that it is of the form

$$F = x_1\bar{p}_1 + x_2\bar{p}_2 + F_n(x_1, \bar{p}_1, x_2, \bar{p}_2) \ , \qquad \text{(III-33)}$$

where F_n is a homogeneous polynomial of degree n. F generates the transformation

$$\bar{x}_i = x_i + \frac{\partial F_n}{\partial p_i} \ , \qquad \text{(III-34a)}$$

$$p_i = \bar{p}_i + \frac{\partial F_n}{\partial x_i} \ . \qquad \text{(III-34b)}$$

If there is to be agreement with Equation (III-32) through first order, it must be the case that

$$\frac{\partial F_n}{\partial \bar{p}_i} = -\psi_i \ , \qquad \text{(III-35a)}$$

$$\frac{\partial F_n}{\partial x_i} = \varphi_i \ . \qquad \text{(III-35b)}$$

Consequently the required generating function can be found if Equation (III-35) is integrable. We shall now show that this is the case.

Compounding the canonical transformations in Equations (III-26) and (III-27), one obtains a canonical transformation which, when truncated, gives Equation (III-32). Let R_i and S_i denote the higher order terms in this process. We may then write the complete canonical transformation as

$$x_i = \bar{x}_i + \psi_i + R_i \ , \qquad \text{(III-36a)}$$

$$p_i = \bar{p}_i + \varphi_i + S_i \ . \qquad \text{(III-36b)}$$

Canonical transformations leave the Poisson bracket relations invariant, or

$$\frac{\partial x_i}{\partial \bar{x}_1}\frac{\partial p_j}{\partial \bar{p}_1} - \frac{\partial x_i}{\partial \bar{p}_1}\frac{\partial p_j}{\partial \bar{x}_1} + \frac{\partial x_i}{\partial \bar{x}_2}\frac{\partial p_j}{\partial \bar{p}_2} - \frac{\partial x_i}{\partial \bar{p}_2}\frac{\partial p_j}{\partial \bar{x}_2} = \delta_{ij} \ , \qquad \text{(III-37a)}$$

$$\frac{\partial x_i}{\partial \bar{x}_1}\frac{\partial x_j}{\partial \bar{p}_1} - \frac{\partial x_i}{\partial \bar{x}_1}\frac{\partial x_j}{\partial \bar{x}_1} + \frac{\partial x_i}{\partial \bar{x}_2}\frac{\partial x_j}{\partial \bar{p}_2} - \frac{\partial x_i}{\partial \bar{p}_2}\frac{\partial x_j}{\partial \bar{x}_2} = 0 \ , \qquad \text{(III-37b)}$$

$$\frac{\partial p_i}{\partial \bar{x}_1}\frac{\partial p_j}{\partial \bar{p}_1} - \frac{\partial p_i}{\partial \bar{p}_1}\frac{\partial p_j}{\partial \bar{x}_1} + \frac{\partial p_i}{\partial \bar{x}_2}\frac{\partial p_j}{\partial \bar{p}_2} - \frac{\partial p_i}{\partial \bar{p}_2}\frac{\partial p_j}{\partial \bar{x}_2} = 0 \ . \qquad \text{(III-37c)}$$

If we substitute Equation (III-36) into these expressions and require that these equations be satisfied order by order, we find that

$$\frac{\partial \varphi_i}{\partial \bar{p}_j} + \frac{\partial \psi_i}{\partial \bar{x}_j} = 0 \; ; \quad i,j = 1,2; \tag{III-38a}$$

$$\frac{\partial \varphi_2}{\partial \bar{x}_1} = \frac{\partial \varphi_1}{\partial \bar{x}_2} \; , \tag{III-38b}$$

$$\frac{\partial \psi_2}{\partial \bar{p}_1} = \frac{\partial \psi_1}{\partial \bar{p}_2} \; . \tag{III-38c}$$

Now a necessary and sufficient condition that Equation (III-35) be integrable is that the Jacobian matrix of the functions φ_1, φ_2, $-\psi_1$, and $-\psi_2$ be symmetric (Wintner 1941). It is easy to verify that in this case the Jacobian matrix is symmetric because of Equation (III-38). Hence Equation (III-35) is integrable, and the required generating function exists. Indeed it may in practice be found by a simple integration of Equation (III-35). Once the generating function is known, the transformation is determined by Equation (III-34). This transformation normalizes H through degree n and repetition of the process may be used to reduce a Hamiltonian to normal form through any order. In the following section we apply the method to a specific example.

CHAPTER IV

NORMAL FORM FOR THE HENON AND HEILES SYSTEM

The normalization process discussed in the previous chapter will now be applied to a concrete problem. Since the Hamiltonian used by Henon and Heiles (Equation II-15) has been extensively studied, we choose it as our example. In passing let us mention that various investigators have studied other Hamiltonians using a different method of expansion for the third integral; these studies have been primarily limited to weak perturbations (Contopoulos 1963, Contopoulos and Moutsoulos 1965). Now whenever a Hamiltonian is reduced to normal form, there are two integrals, namely the energy integral expressed in normal form and either J_1, say, or the integral given in Equation (III-19), if the linear frequencies are equal. Laying aside all questions of convergence, we can calculate the level curves in any plane using these two integrals. However these level curves are expressed in the transformed coordinate system and therefore cannot be compared with the results obtained by direct numerical integration of the original coordinate equations of motion. Using the inverses of the transformations employed in the normalization however, it is possible to express the third integral in terms of the original coordinates. Then one can use this integral along with the original Hamiltonian to calculate the level curves in the original coordinates system. These results may then be easily compared with the published data, and we can gain some idea as to the accuracy of the truncated integral. It is of

interest to calculate sets of level curves in both the original and transformed coordinates in order to see the effects of the transformation.

An enormous amount of computational work is needed to normalize a given Hamiltonian to any reasonably high order. A series of computer programs were written to handle the algebra in this process. The major problem involves writing an efficient subroutine for multiplying two polynomials in four variables. In our approach, we used a four subscript matrix representation for the polynomials where location in the array represents the powers of the variables, and the number stored in the array location is the corresponding coefficient. For example, suppose that it is desired to store a polynomial of maximum degree eight in the array X. Then the term $xq_1^m q_2^n p_1^s p_2^t$ would be represented in this array by storing the number x in the array position (m,n,s,t). There are a number of more sophisticated methods for computer representation of algebraic expressions (for a review of the literature see Davis, 1968). Using our storage methods, polynomial multiplication is reduced to relatively simple manipulations with four dimensional arrays. Our method would require around 50,000 words of computer memory if executed by brute force. However, there are a number of techniques for reducing the computer requirements by half, such as using triangular arrays. Nonetheless, eighth-order polynomial manipulation appears to be the limit for most medium-sized computers. For our problem, the programming is further simplified by the fact that all the coefficients are either purely real or purely imaginary. In addition it turns out never to be necessary to compute either of the two generating functions. All that is needed is a knowledge of Equations

(III-7) and (III-34) which arises naturally during the computations.

The actual calculations were done in a sequence of stages on a Burroughs B-5500 computer, and all of the programs were written in Algol. The following sequence of programs was used:

(1) A Hamiltonian was read into the computer along with two integers, one specifying the degree through which normalization was to be performed and one giving the degree of the terms which were to be normalized. A subroutine converted the Hamiltonian into complex form, and another subroutine calculated the coefficients in Equation (III-7). A third subroutine transformed these equations back to real form. A series of iterations then determined Q_1, P_1, Q_2, and P_2 as a power series in $\bar{Q}_1$, $\bar{P}_1$, $\bar{Q}_2$ and $\bar{P}_2$ through the desired order. A typical run time for this section of the program was about ninety seconds.

(2) The same Hamiltonian used in step one was then read back into the computer along with the transformation from step one. The transformed Hamiltonian was then calculated and stored on magnetic tape or on punched cards for future use. Run times vary for this program, depending upon the Hamiltonian; typical values range from 200 to 900 seconds.

The output from stage two is then fed back into stage one, and the entire process is repeated until normalization through the order desired is achieved. For larger computer installations both programs could easily be combined into one.

The results for the Henon and Heiles Hamiltonian in real normal form

through eighth order are given in Table 1. These same results were obtained independently by Gustavson in 1966. In Table 1 the exponents (m,n,s,t) for the variables are listed in a column to the left of the corresponding coefficient of $\bar{x}_1^m \bar{x}_2^n \bar{p}_1^s \bar{p}_2^t$. The elements with zero coefficients have been suppressed. Table 2 lists the same Hamiltonian in normal form in terms of complex coordinates, where exponent (m,n,s,t) corresponds to $\bar{Q}_1^m \bar{Q}_2^n (i\bar{P}_1)^s (i\bar{P}_2)^t$.

The integral in the "normal form" set of coordinates is simply the sum of the actions, $J_1 + J_2$. By applying the inverse of each transformation in succession, it is possible to obtain a truncated power series for the integral in the original coordinates. The inversion of each transformation is performed using a minor modification of the program listed as (1) on the previous page. Our results for the integral in the original Cartesian coordinates are shown in Table 3.

Level Curves for the Henon and Heiles System

We first find the approximate level curves for the Henon and Heiles system in the transformed coordinate system. The normalized Hamiltonian through eighth order will be denoted by H_n. Then for fixed energy

$$H_n(\bar{x}_1, \bar{p}_1, \bar{x}_2, \bar{p}_2) = E \ . \qquad \text{(IV-1)}$$

The integral in these coordinates is, using Equation (III-19) in Cartesian coordinates,

$$\frac{\bar{p}_1^2 + \bar{x}_1^2 + \bar{p}_2^2 + \bar{x}_2^2}{2} = I_o \qquad \text{(IV-2)}$$

Table 1. Normal Form for the Henon and Heiles System

(Cartesian Coordinates)

Order	Exponent	Coefficient	Exponent	Coefficient
2	0002	.500000	0020	.500000
	0200	.500000	2000	.500000
4	0004	-0.104167	0040	-0.104167
	0400	-0.104167	4000	-0.104167
	0022	-0.208333	0202	-0.208333
	2002	0.375000	2020	-0.208333
	2200	-0.208333	0220	0.375000
	1111	-1.166667		
6	0006	-0.067998	0060	0.029225
	0600	-0.067998	6000	0.029225
	0042	-0.787326	0420	0.063368
	0204	-0.203993	0240	-0.228299
	0024	0.671007	0402	-0.203993
	4020	0.087674	2004	0.063368
	2040	0.087674	4002	-0.228299
	2400	0.671007	4200	-0.787326
	1311	1.215278	1131	-1.118055
	3111	-1.118055	1113	1.215278
	2220	-1.015625	2202	0.734375
	0222	0.734375	2022	-1.015625
8	0008	-0.077532	0080	0.043590
	0800	-0.077532	8000	0.043590
	0044	-0.101825	0404	-0.465193
	0440	2.126274	4400	-0.101825
	4004	2.126274	4040	0.261543
	0422	-0.588219	0242	-1.678323
	2042	-2.757827	2420	-1.678322
	2204	-0.588219	2240	-2.768427
	0224	0.875854	2024	-1.678323
	2404	0.875854	4022	-2.768427
	4220	-2.757827	4202	-1.678323
	2600	0.779975	0206	-0.310129
	0620	-0.684097	0026	0.779975
	0062	-0.915742	0260	-0.926342
	0602	-0.310129	2060	0.174362
	6200	-0.915742	6020	0.174362
	6002	-0.926342	2222	18.859 82
	2006	-0.684097	1133	2.949344

Table 1. (Continued)

Order	Exponent	Coefficient	Exponent	Coefficient
8	1511	2.928144	1115	2.928144
	1151	0.021200	5111	0.021200
	3113	-11.861741	1331	-11.861741
	3131	0.042401	1313	5.856289
	3311	2.949344		

Table 2. Normal Form for the Henon and Heiles System

(Complex Coordinates)

Order	Exponent	Coefficient	Exponent	Coefficient
2	1010	1.000000	0101	1.000000
4	0202	-0.416667	0220	-0.583333
	1111	0.333333	2002	-0.583333
	2020	-0.416667		
6	0303	-0.543981	0321	1.215278
	1212	2.987500	1230	-1.118056
	2103	1.215278	2121	-4.062500
	3012	-1.118056	3030	0.233796
8	0404	-1.240516	0422	5.856289
	0440	3.702771	1313	0.767028
	1331	-8.912397	2204	5.856289
	2222	8.790046	2240	0.042400
	3113	-8.912397	3131	-14.736677
	4004	3.702771	4022	0.042400
	4040	0.697447		

Table 3. Integral for the Henon and Heiles System

(Cartesian Coordinates)

Order	Exponent	Coefficient	Exponent	Coefficient
2	0002	0.500000	0020	0.500000
	0200	0.500000	2000	0.500000
3	2100	1.000000	0300	-0.333333
4	0004	0.104167	0040	0.104167
	0400	0.104167	4000	0.104167
	1111	1.166667	0202	0.208333
	2002	-0.375000	0022	0.208333
	2200	0.208333	0220	-0.375000
	2020	0.208333		
5	0122	2.333333	1211	1.166667
	2102	-1.916667	2120	-0.750000
	0500	-0.138889	1031	0.777778
	1013	-2.333333	3011	1.166667
	0320	1.027778	2300	0.277778
	0302	-0.138889	0140	-0.777778
	4100	0.416667		
6	0006	0.111400	0060	-0.245081
	0600	0.157697	6000	0.014178
	0042	2.473090	0420	1.537326
	4020	-0.540799	0204	0.334201
	0240	2.448785	2004	1.379340
	2040	-0.735243	0024	-2.874132
	0402	0.334201	4002	-0.662326
	2400	-0.818576	4200	1.334201
	1311	-1.506944	1131	0.048611
	3111	5.493055	1113	-8.506944
	2220	1.421875	2202	3.828125
	0222	-1.494792	2022	4.921875
7	0700	-0.222801	0160	0.926852
	6100	0.668403	0520	0.500694
	1015	2.780555	1051	-0.926852
	5011	1.993055	0502	-0.445602
	2500	0.222801	0142	-1.853704
	0304	-0.222801	0340	-2.825116
	0124	-2.780555	2104	3.448958
	2140	0.846644	1411	4.086574
	4102	0.490972	4300	1.114005

Table 3. Integral for the Henon and Heiles System

(Cartesian Coordinates)

Order	Exponent	Coefficient	Exponent	Coefficient
7	4120	-0.656250	1231	-0.178241
	2122	9.143749	2320	4.031481
	1213	-5.026388	0322	1.800231
	2302	-4.141666	3031	-0.178241
	3013	-5.026389	3211	-2.294444
8	0008	0.170969	0026	0.758204
	0044	0.041032	0062	2.325175
	0080	-0.157291	0206	0.683875
	0224	-6.816321	0242	4.937312
	0260	2.489305	0404	1.025813
	0422	6.724503	0440	-6.318201
	0602	0.832409	0620	2.533865
	0800	0.31 502	1115	-8.206501
	1133	-8.534761	1151	-0.328260
	1313	-26.608372	1331	37.679102
	1511	-5.354649	2006	1.832785
	2024	4.349445	2042	7.139656
	2060	-0.629165	2204	4.175661
	2222	-70.817274	2240	13.328323
	2402	-8.661386	2420	-5.008193
	2600	-1.185357	3113	44.476016
	3131	-8.998186	3311	-8.106983
	4004	-5.468586	4022	10.779480
	4040	-1.175460	4202	13.172362
	4220	12.772063	4400	1.095894
	5111	-2.752334	6002	1.538688
	6020	-0.813347	6200	2.646548
	8000	0.045795		

where I_o is a constant. We choose to find level curves in the $x_2 - p_2$ plane by setting $\bar{x}_1$ equal to zero and solving Equation (IV-2) for $\bar{p}_1$. Obviously

$$p_1 = \sqrt{2I_o - \bar{x}_2^2 - \bar{p}_2^2} \qquad \text{(IV-3)}$$

Following the convention in Chapter II, the positive branch has been taken in Equation (IV-3). The expression for $\bar{p}_1$ is now substituted into Equation (IV-1) yielding

$$H_n(0, \sqrt{2I_o - \bar{x}_2^2 - \bar{p}_2^2}, \bar{x}_2, \bar{p}_2) = E \; . \qquad \text{(IV-4)}$$

The result is an eighth order polynomial in $\bar{x}_2$ and $\bar{p}_2$ since $\bar{p}_1$ enters only in even powers for this particular normal form.

The level curves were calculated numerically to four figure accuracy using the method of secants. Certain points in the plane were selected, and values of I_o for these points were calculated from Equation (IV-2). $\bar{x}_2$ was then fixed, and values of $\bar{p}_2$ satisfying Equation (IV-4) were then found. The value of $\bar{x}_2$ was then incremented, and the process repeated. The results obtained for E = 1/12 are shown in Figure 6. The separatrix in this coordinate system is the curve passing through the origin. There are four stable periodic solutions, two on each axis. There are also three unstable periodic solutions on the separatrix. The level curves for E = 1/6, the disassociation energy are shown in Figure 7. There are still four obvious stable periodic solutions, but the separatrix has split on the $\bar{p}_2$ axis. Presumably there are level curves encircling the separatrix,

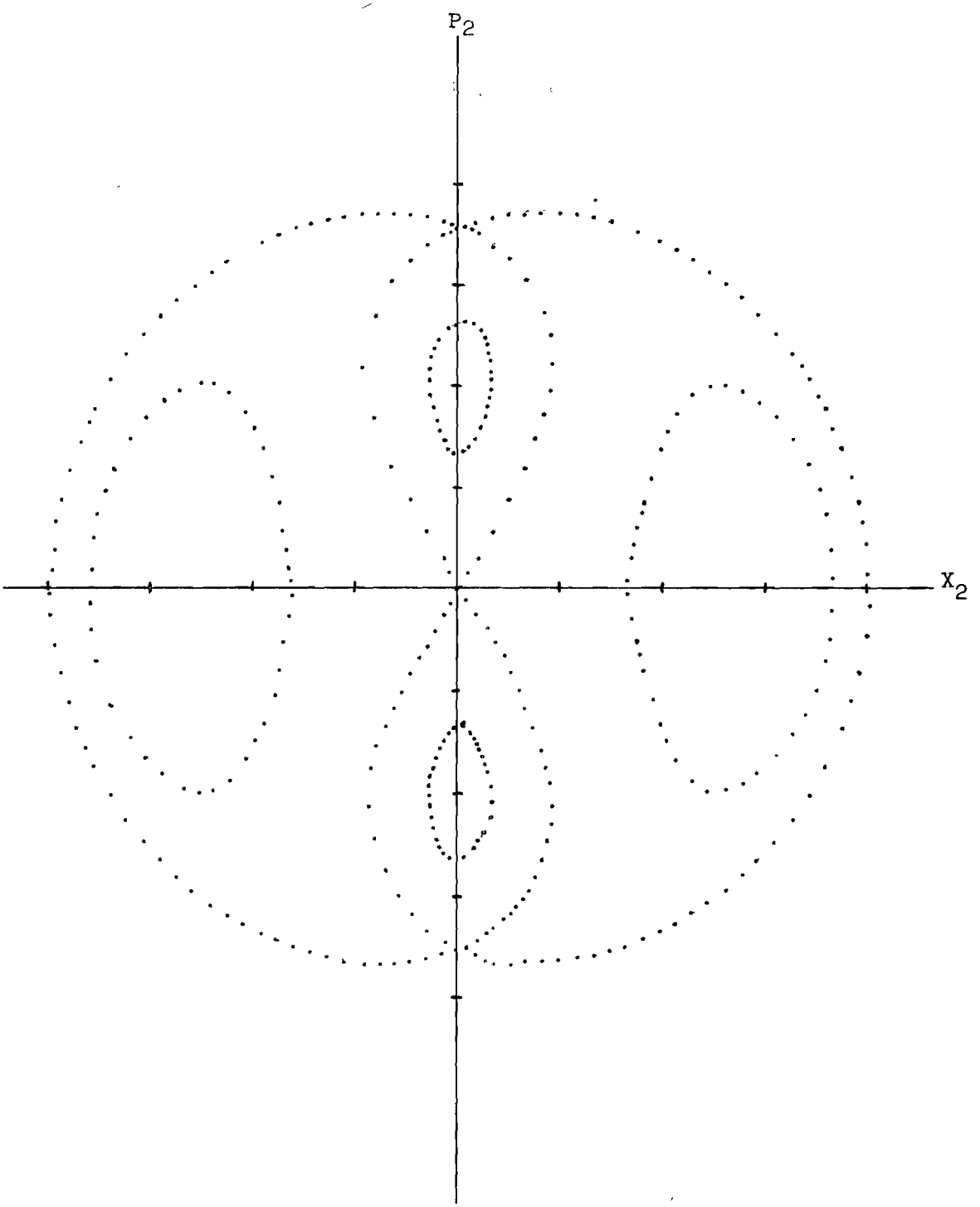

Figure 6. Level Curves for the Henon and Heiles System Normalized Through Eighth Order. Energy E = 1/12. (Transformed Coordinates)

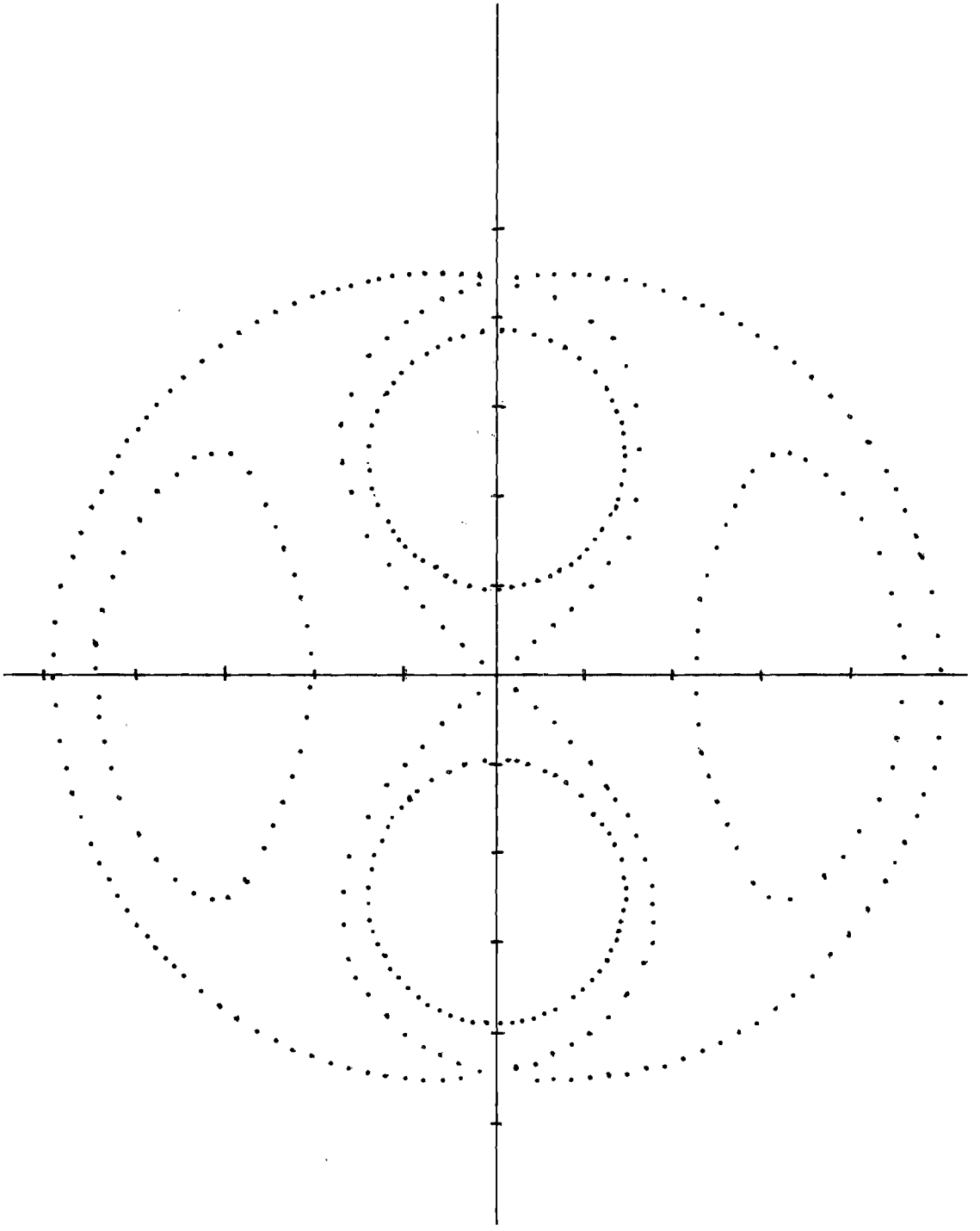

Figure 7. Level Curves for the Henon and Heiles System Normalized Through Eighth Order. Energy E = 1/6. (Transformed Coordinates)

passing through the gap on the $\bar{p}_2$ axis and encircling the stable periodic solutions on the $\bar{p}_2$ axis. The level curve pattern here is somewhat more complicated than in Figure 6.

A similar procedure may be applied to find the level curves in the original coordinate system. The integral is now an eighth degree polynomial and the energy is given by Equation (II-15). Level curves in the x_2 - p_2 plane are found by setting x_1 equal to zero in both the integral and the Hamiltonian. The Hamiltonian is solved for p_2 and the resulting expression is substituted into the integral. The level curve is then obtained by solving for p_2 as a function of x_2.

The level curves for the Henon and Heiles system, using the truncated eighth order integral, for E = 1/12 in the original coordinates are shown in Figure 8. There is impressive agreement between the level curves of Figure 8 and those calculated by direct integration of the equations of motion shown in Figure 1. The agreement is, as one would expect, even better at lower energies (Gustavson 1966). Comparing Figure 6 with Figure 8, we note that the change from the original to the normalized set of coordinates serves only to shift the level curves slightly. At E = 1/8, shown in Figure 9, the level curves obtained using the third integral have become much more complicated. Comparing the curves of Figure 9 with the directly integrated curves of Figure 4, we note only a partial agreement. The four stable periodic solutions are in approximately the correct location and they are still surrounded by closed level curves. It is possible that if the normalization were carried to higher order better agreement could be obtained. However, in Figure 4 the separatrix has been destroyed and there are a number of initial conditions which give points that seem

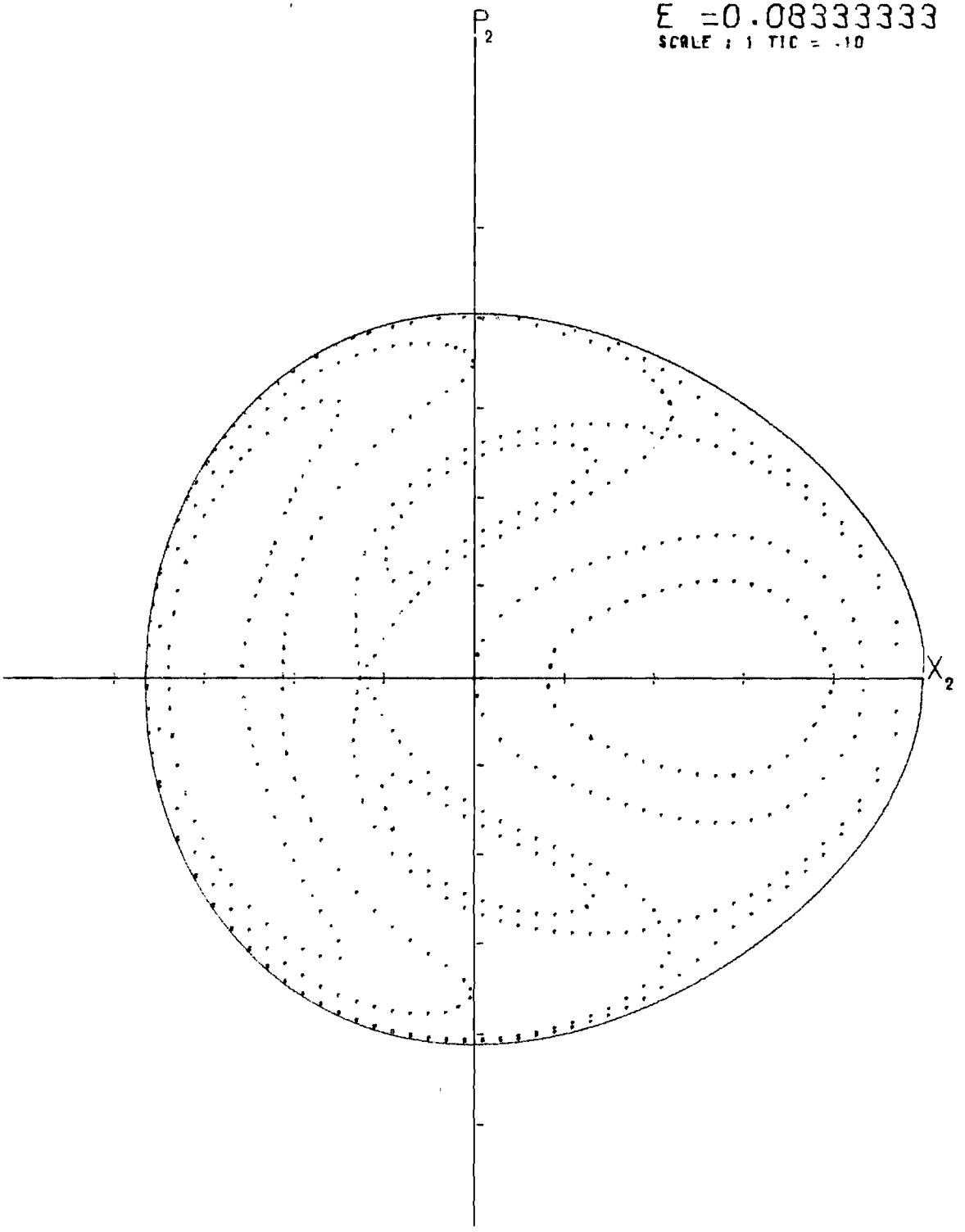

Figure 8. Theoretical Level Curves for the Henon and Heiles System. Energy E = 1/12.

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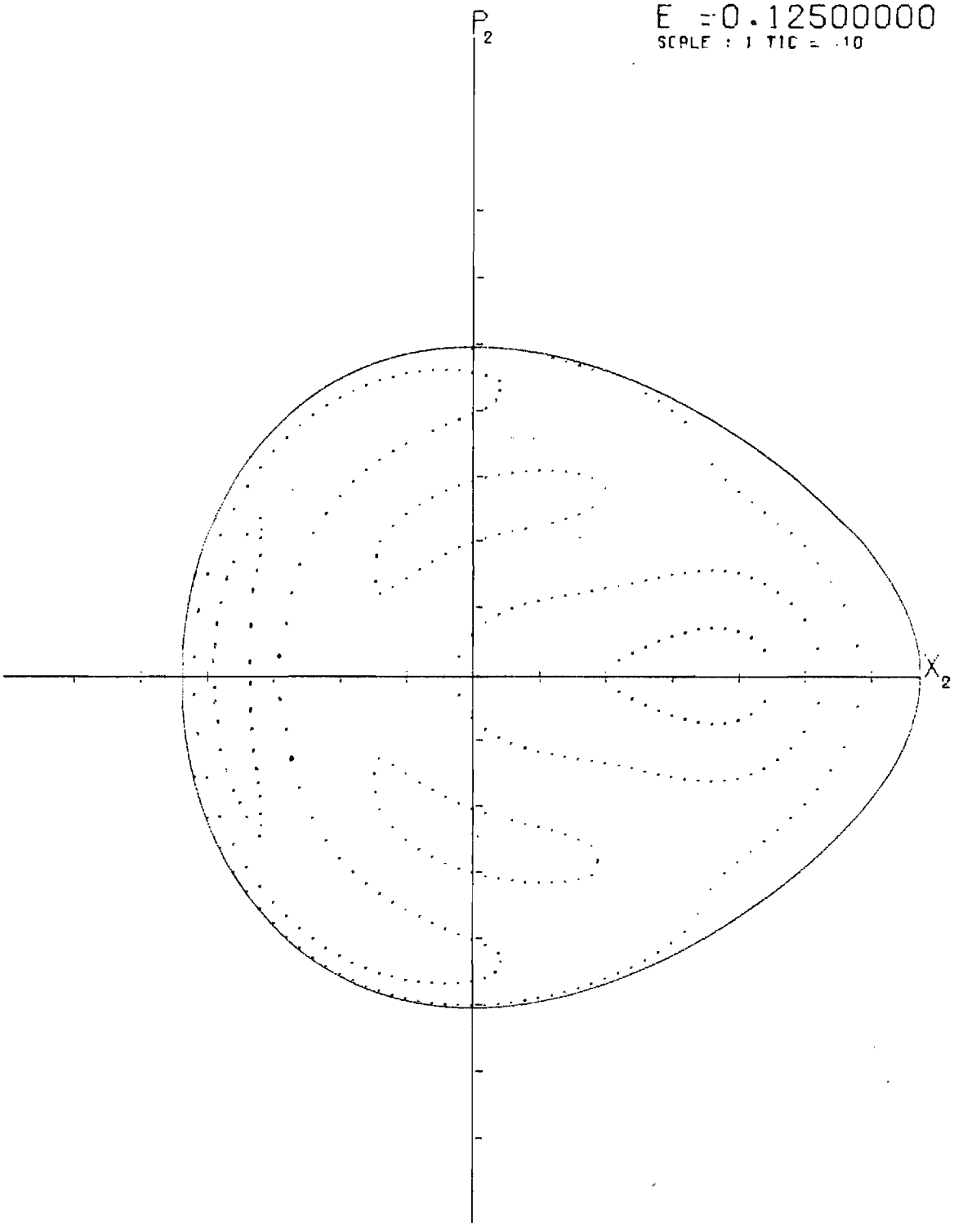

Figure 9. Theoretical Level Curves for the Henon and Heiles System. Energy E = 1/8.

to lie on no regular curve. It is difficult to believe that the addition of higher order terms could improve the agreement in this region, for reasons we shall discuss later. The departure between the level curves calculated using the third integral and those calculated directly is found to increase with increasing energy until at E = 1/6 (dissociation), there is almost no agreement.

An interesting feature of Figure 4 which has no counterpart in the level curves calculated from the truncated integral is the chain of five islands about the periodic solution on the positive x_2 axis. The question naturally arises as to whether or not higher order terms in the third integral can account for island chains. This brings up some points that will be needed in later chapters, so we shall discuss the matter in some detail here. In order to illustrate the small denominator difficulty hidden in our modified Birkhoff technique and to show that this technique cannot lead to the chain of five islands in Figure 4, let us consider the integrable Hamiltonian

$$H = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2 + \beta J_1^{3/2} J_2 \cos(3\varphi_1 - 2\varphi_2), \quad \text{(IV-5)}$$

where J_i and φ_i are action-angle variables and β is a constant. Hamiltonian (IV-5) is certainly integrable since it possesses the isolating third integral

$$\bar{I} = 2J_1 + 3J_2 , \quad \text{(IV-6)}$$

where $\bar{I}$ is a constant. Moreover, the level curves, shown in Figure 11, calculated for this Hamiltonian exhibit a chain of three islands.

Now with Hamiltonian (IV-5), the modified Birkhoff technique would use

$$H_o = J_1 + J_2 \qquad \text{(IV-7)}$$

as the unperturbed Hamiltonian and

$$\omega_1 = \frac{\partial H_o}{\partial J_1} = \omega_2 = \frac{\partial H_o}{\partial J_2} = 1 \qquad \text{(IV-8)}$$

as the unperturbed frequencies. One would then seek to eliminate the angle-dependent term in the Hamiltonian (IV-5) using the canonical transformation generated by

$$K = \bar{J}_1\varphi_1 + \bar{J}_2\varphi_2 + \gamma\, \bar{J}_1^{3/2}\bar{J}_2 \sin(3\varphi_1 - 2\varphi_2) \qquad \text{(IV-9)}$$

where $\bar{J}_1$ and $\bar{J}_2$ are the new action variables and γ is a constant to be determined. Through first order, the resulting transformation equations are given by

$$J_1 = \bar{J}_1 + 3\gamma\bar{J}_1^{3/2}\bar{J}_2 \cos(3\theta_1 - 2\theta_2)\,, \qquad \text{(IV-10a)}$$

$$J_2 = \bar{J}_2 - 2\gamma\bar{J}_1^{3/2}\bar{J}_2 \cos(3\theta_1 - 2\theta_2)\,, \qquad \text{(IV-10b)}$$

$$\varphi_1 = \theta_1 - \frac{3}{2}\gamma\, \bar{J}_1^{1/2}\bar{J}_2 \sin(3\theta_1 - 2\theta_2)\,, \qquad \text{(IV-10c)}$$

$$\varphi_2 = \theta_2 - \gamma\bar{J}_1^{3/2} \sin(3\theta_1 - 2\theta_2)\,. \qquad \text{(IV-10d)}$$

where θ_1 and θ_2 are the new angle variables conjugate to $\bar{J}_1$ and $\bar{J}_2$. Putting Equations (IV-10) into Hamiltonian (IV-5), we find that the angle

dependent term with coefficient β can be eliminated provided we set

$$\gamma = - \frac{\beta}{3\omega_1 - 2\omega_2} = -\beta \ , \qquad \text{(IV-11)}$$

Of course, if all orders generated by the transformation (IV-9) were substituted into Hamiltonian (IV-5), one would obtain new, higher-order terms which would have to be sequentially eliminated, but here we are not interested in continuing the process. Rather let us note that the denominator in Equation (IV-11) is independent of initial conditions and can never be smaller than unity.

Now let us consider Kolmogorov's process for eliminating the angle dependent term in Hamiltonian (IV-5) Kolmogorov uses

$$H_o = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2 \qquad \text{(IV-12)}$$

as the unperturbed Hamiltonian and

$$\Omega_1(J_1, J_2) = \frac{\partial H_o}{\partial J_1} = 1 - 2J_1 - 3J_2 \ , \qquad \text{(IV-13a)}$$

$$\Omega_2(J_1, J_2) = \frac{\partial H_o}{\partial J_2} = 1 - 3J_1 + 2J_2 \qquad \text{(IV-13b)}$$

as the unperturbed frequencies. At this point, one again uses the same Transformation (IV-9) as was used before. Upon substituting Equations (IV-10) into Hamiltonian (IV-5), one may choose to eliminate the angle dependent term with coefficient β by requiring that

$$\gamma = \frac{\beta}{3\Omega_1 - 2\Omega_2} = - \frac{\beta}{(1 - 13\bar{J}_2)} \qquad \text{(IV-14)}$$

The transformed Hamiltonian, retaining only lowest order terms, is

$$H = \bar{J}_1 + \bar{J}_2 - \bar{J}_1^2 - 3\bar{J}_1\bar{J}_2 + \bar{J}_2^2 . \qquad \text{(IV-15)}$$

Thus, here the frequency denominator in Equation (IV-14) does depend on initial conditions through the constant $\bar{J}_2$; moreover, the denominator can be zero for $\bar{J}_2 = 1/13$. As a consequence, Kolmogorov's technique cannot eliminate the angle dependent term in Hamiltonian (IV-5) when $\bar{J}_2 \simeq 1/13$.

We now observe that, roughly speaking, the modified Birkhoff technique can be obtained from Kolmogorov's method by expanding the Kolmogorov denominators in powers of the action. Equation (IV-14), for example, becomes

$$\gamma = \left[\frac{-\beta}{3\omega_1 - 2\omega_2}\right] + \ldots , \qquad \text{(IV-16)}$$

where the dots denote a power series in the action variables. If we retain only the lowest order lead term, we obtain the modified Birkhoff Equation (IV-11). Now this expansion is permissible as long as $\bar{J}_2 \neq 1/13$; but when $\bar{J}_2 \simeq 1/13$, the modified Birkhoff technique represents an impermissible expansion of a small denominator. Finally, since the chain of three islands for Hamiltonian (IV-5) shown in Figure 11 occurs when $\bar{J}_2 = 1/13$, we see that there is no hope of obtaining it using the modified Birkhoff technique of Equations (IV-7) through (IV-11).

In summary then, the situation appears to be as follows. The unmodified Birkhoff technique starts with Hamiltonian (II-4), i.e., with

$$H = \omega_1 J_1 + \omega_2 J_2 + V(J_1, J_2, \varphi_1, \varphi_2) , \qquad \text{(IV-17)}$$

and uses $(\omega_1 J_1 + \omega_2 J_2)$ as the unperturbed Hamiltonian. In eliminating all angle dependent terms using the assumption that ω_1 and ω_2 are incommensurate, Birkhoff is assuming that no resonant terms ever arise from $V(J_1,J_2,\varphi_1,\varphi_2)$ which lead to convergence endangering small denominators; or equivalently, he is assuming that the perturbed motion is on tori close to the unperturbed tori in the sense that the perturbed motion is on tori close to the unperturbed tori in the sense that the perturbed action variables are close to the unperturbed. As Kolmogorov theory shows, this assumption is valid for most initial conditions when $V(J_1,J_2,\varphi_1,\varphi_2)$ is sufficiently small.

However when ω_1 and ω_2 in Hamiltonian (IV-17) are commensurate, then as shown in Chapter III, a single class of angle dependent terms obviously lead to zero denominators and cannot be eliminated. The modified Birkhoff technique, still using $(\omega_1 J_1 + \omega_2 J_2)$ as the unperturbed Hamiltonian, then asserts that the system motion is close to that of the integrable Hamiltonian

$$H = \omega_1 J_1 + \omega_2 J_2 + V(J_1,J_2,\ n\varphi_1 - m\varphi_2) \qquad \text{(IV-18)}$$

Moreover, Kolmogorov theory indicates that, for sufficiently small $V(J_1,J_2,\varphi_1,\varphi_2)$ in Equation (IV-17), the modified Birkhoff technique is valid, again for most initial conditions since for them the only angle dependent, resonant terms which seriously affect the motion are $(n\varphi_1 - m\varphi_2)$ and its harmonics.

In both the above cases, the restriction "for most initial conditions" arises because more than one class of resonant term can influence the motion. We indicate this fact by observing that Kolmogorov theory

treats the general Hamiltonian (II-5), i.e.,

$$H = H_o(J_1,J_2) + V(J_1,J_2,\Phi_1,\Phi_2) \qquad \text{(IV-19)}$$

using $H_o(J_1,J_2)$, $\Omega_1 = \frac{\partial H_o}{\partial J_1}$, and $\Omega_2 = \frac{\partial H_o}{\partial J_2}$ as the unperturbed system. This method then encounters resonant, angle dependent terms leading to small or zero denominators whenever $n\Omega_1(J_1,J_2) = m\Omega_2(J_1,J_2)$. Since this resonant condition $n\Omega_1 = m\Omega_2$ depends on initial conditions, one set of initial conditions can involve the resonance $r\Omega_1 = s\Omega_2$ and its harmonics while another set of initial conditions may involve a completely different resonance.

As a consequence, it becomes clear that the modified Birkhoff method fails because it represents an improper expansion in those regions for which an additional resonance, other than the basic resonance $n\omega_1 = m\Omega_2$ in Hamiltonian (IV-17), is dominant. As demonstrated by the example Hamiltonian (IV-5), this additional resonance manifests itself in the level curves as a chain of islands. Thus one sees that the chain of islands in Figure 4 indicates the presence of a dominant resonance, other than the $\omega_1 = \omega_2 = 1$ resonance used in the modified Birkhoff technique. In the final Chapter, we shall return to this point and indicate how one might predict this chain of islands for the Henon and Heiles system. For the moment however, it will be more informative to initiate the next Chapter which investigates the properties of island chains generated by integrable Hamiltonians.

CHAPTER V

THE EFFECTS OF RESONANCE TERMS

In Chapter IV it was seen that the theory developed thus far cannot account for some of the features observed in the directly integrated level curves for the Henon-Heiles Hamiltonian. The modified Birkhoff technique, which predicts the level curves quite accurately at small or moderate energies, fails completely to predict the sudden appearance both of chains of islands and the seemingly ergodic trajectories which cover an increasing region of phase space as the system approaches dissociation. The question then arises as to what elements in the Henon-Heiles Hamiltonian cause the sudden appearance and growth of these irregularities. In order to gain insight into this question of amplitude instability, we begin in this chapter to study several model Hamiltonians. No claim is made concerning the physical reality of any of these models; they were chosen for their convenience in demonstrating typical behavior for conservative dynamical systems with two degrees of freedom. It is reasonable to suppose that the behavior exhibited by our examples is also exhibited by most physical potentials. In this chapter we determine the source of island chains, and in the following chapter we establish the origin of amplitude instability.

The first model that we shall consider is the angle independent Hamiltonian

$$H = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2. \qquad (V\text{-}1)$$

The terms $J_1 + J_2$ in this Hamiltonian describe two linear oscillators with equal frequencies. The remaining nonlinear, quadratic terms couple the oscillators together to the extent of providing nonlinear frequency shifts useful in later work. From Equations (II-7), (II-8), and (II-3) we have that both J_1 and J_2 are constant, and that the frequencies Ω_1 and Ω_2 are given by

$$\Omega_1 = 1 - 2J_1 - 3J_2 , \qquad \text{(V-2a)}$$

$$\Omega_2 = 1 - 3J_1 + 2J_2 , \qquad \text{(V-2b)}$$

and that

$$J_1 = \frac{1}{2} (p_1^2 + x_1^2) , \qquad \text{(V-3a)}$$

$$J_2 = \frac{1}{2} (p_2^2 + x_2^2) , \qquad \text{(V-3b)}$$

Since neither x_1 nor p_1 appear in Equation (V-3b), we have that the level curves for Hamiltonian (V-1) in the $x_2 - p_2$ plane (henceforth called the J_2 plane) are especially simple, being concentric circles centered on the origin. We shall later have need for level curves in the $x_1 - p_1$ plane (henceforth called the J_1 plane) defined by $x_1 - p_1$ points for which $x_2 = 0$ and $p_2 \geq 0$. For Hamiltonian (V-1) the level curves in the J_1-plane are again concentric circles centered on the origin. From Equation (V-3) we have that $J_1 \geq 0$ and $J_2 \geq 0$. Thus if Hamiltonian (V-1) were purely linear in the J's, we would have $H \geq 0$. Moreover for fixed energy E, we would have a largest circular level curve, called the bounding level curve, in both the J_1 - and J_2-planes, given by $J_1 = E$ or $J_2 = E$. The quadratic

terms in Hamiltonian (V-1) complicate the situation to the extent of allowing two branches for J_1 and J_2 and allowing $E \leq 0$. We eliminate the physically uninteresting branch by requiring that $0 \leq E \leq 3/13$ and by always choosing the root for J_1 or J_2 which goes to zero as E tends to zero. With these restrictions the level curves for Hamiltonian (V-1) in either the J_1- or J_2-plane are concentric circles lying on the boundary or inside an area enclosed by a bounding level curve.

Let us now recall from the discussion around Equation (II-10) that the system motion for Hamiltonian (V-1) may be represented as motion on tori with frequencies Ω_1 and Ω_2 given by Equation (V-2). As a consequence the concentric level curves in either the J_1- or J_2-plane may be thought of as the intersections of either plane with the tori bearing the system motion. Hence we may designate a level curve by specifying the frequencies on the associated torus. Since we are going to be concerned with changes in these tori and their associated level curves due to the influence of certain simple perturbation terms, we choose here to catalogue the positions of the three unperturbed level curves $\Omega_1 = \Omega_2$, $3\Omega_1 = 2\Omega_2$, $2\Omega_1 = 3\Omega_2$ which will be most affected by the perturbations.

First we find the position of the $\Omega_1 = \Omega_2$ level curves. Here Equation (V-2) implies that $J_1 = 5\ J_2$. Using Equation (V-1), we then find

$$J_1 = \frac{5}{13}\left[1 - \sqrt{1 - \frac{13}{3}E}\ \right] \qquad \text{(V-4a)}$$

and

$$J_2 = \frac{1}{13}\left[1 - \sqrt{1 - \frac{13}{3}E}\ \right] \qquad \text{(V-4b)}$$

as the values for J_1 and J_2 on the $\Omega_1 = \Omega_2$ level curves in the J_1- and J_2-planes respectively. Here we have chosen the branch for J_1 and J_2 which goes to zero as E tends to zero, and we note that J_1 and J_2 will be real and positive for $0 \leq E \leq 3/13$. From Equation (V-4) we see that the $\Omega_1 = \Omega_2$ level curves always exist for $0 \leq E \leq 3/13$ and that the radii of the $\Omega_1 = \Omega_2$ circular level curves increase monotonically with energy.

Next we consider the $3\Omega_1 = 2\Omega_2$ level curves. Using Equation (V-2), we find that $J_2 = 1/13$ for all values of J_1. Putting $J_2 = 1/13$ into Equation (V-1) and making the previously mentioned branch choice, we find that J_1 is given by

$$J_1 = \frac{5}{13} - \sqrt{\frac{3}{13} - E} \quad , \qquad \text{(V-5)}$$

In order for J_1 to be positive in Equation (V-5), we must have $E \geq 14/169 \simeq .083$. At $E = 14/169$, $J_1 = 0$. Thus the $3\Omega_1 = 2\Omega_2$ level curve in the J_1-plane appears abruptly at the origin and then moves outward with increasing energy E. In the J_2-plane the $3\Omega_1 = 2\Omega_2$ level curve is fixed at a radius corresponding to $J_2 = 1/13$. However the $3\Omega_1 = 2\Omega_2$ level curve at this position does not lie on or within the bounding level curve $J_2^{(B)}$, given by

$$J_2^{(B)} = \frac{1}{2}\left(-1 + \sqrt{1 + 4E}\right), \qquad \text{(V-6)}$$

until $E = 14/169$. Hence the $3\Omega_1 = 2\Omega_2$ level curve appears abruptly in the J_2-plane at the bounding level curve rather than at the origin.

Finally we consider the $2\Omega_1 = 3\Omega_2$ level curves. Since the algebraic

expressions for J_1 and J_2 on these level curves have cumbersome coefficients, we shall not reproduce these expressions here. Rather, we note that the relationship between J_1 and J_2 found using Equation (V-2) is

$$J_1 = (1 + 12J_2)/5 \ . \qquad \text{(V-7)}$$

Thus we immediately have that the smallest allowed value of J_1 is 1/5, occurring when $J_2 = 0$. As a consequence the $2\Omega_1 = 3\Omega_2$ level curve appears abruptly at the origin of the J_2-plane when $E = .16$ and then moves outward. The energy of appearance is found by setting $J_1 = 1/5$ and $J_2 = 0$ in Equation (V-1). In the J_1-plane the $2\Omega_1 = 3\Omega_2$ level curve appears abruptly at the bounding level curve $J_1^{(B)}$, given by

$$J_1^{(B)} = \frac{1}{2}\left(1 - \sqrt{1 - 4E}\right), \qquad \text{(V-8)}$$

for $E = 0.16$ and $J_1 = 1/5$.

These results are fairly typical for a wide class of angle independent Hamiltonians. If the linear terms in the Hamiltonian are $rJ_1 + sJ_2$, then the torus $s\Omega_1 = r\Omega_2$ exists for all allowed energies while the remaining rational tori $t\Omega_1 = u\Omega_2$ appear abruptly for specific $E > 0$ either at the origin or the bounding level curve. For the case of interest here, the $\Omega_1 = \Omega_2$ level curves exist at all energies while the rational tori $m\Omega_1 = n\Omega_2$ abruptly appear only for specific $E > 0$. In particular, we find that the closer the ratio (m/n) is to unity, the lower the energy of appearance. This is quite plausible since energy is required to produce frequency shifts away from the unperturbed ratio of unity.

We shall now study the effects of various perturbations on

Hamiltonian (V-1). It is possible to add a wide class of perturbations to any angle independent Hamiltonian and still have an integrable system. Consider, for example, an unperturbed Hamiltonian $H_o(J_1, J_2)$ and a perturbation of the form $V(m\varphi_1 - n\varphi_2, J_1, J_2)$. The resulting Hamiltonian,

$$H = H_o(J_1, J_2) + V(m\varphi_1 - n\varphi_2, J_1, J_2), \qquad (V\text{-}9)$$

is integrable since there is another integral independent of the energy, namely $\Phi = nJ_1 + mJ_2$. That Φ is an integral can easily be verified using Hamilton's equations. The perturbation $V(m\varphi_1 - n\varphi_2, J_1, J_2)$ may be expanded in a Fourier series involving the harmonics of $(m\varphi_1 - n\varphi_2)$. Thus in the study of integrability preserving perturbations, functions proportional to $\cos(m\varphi_1 - n\varphi_2)$ or to $\sin(m\varphi_1 - n\varphi_2)$ play a crucial role. Single perturbation terms of this type will be called an "m-n resonance." The word "resonance" appears because these terms cause energy sharing between unperturbed oscillators as is evidenced by the appearance of small denominators in the Kolmogorov theory briefly mentioned in Chapter IV. Whenever a single resonance term such as $\cos(m\varphi_1 - n\varphi_2)$ is used as a perturbation, it is multiplied by some function of the action variables. For our purposes, we need only require that this function $f(J_1, J_2)$ $\cos(m\varphi_1 - n\varphi_2)$ be a polynomial in $x_i - p_i$ upon substitution of transformation (II-3). There are an infinity of functions $f(J_1, J_2)$ and we shall always choose the one of lowest order. We shall now study the effects of adding a single resonant term to Hamiltonian (V-1).

First the effect of a 2-2 resonance term of the form $\alpha J_1 J_2 \cos 2(\varphi_1 - \varphi_2)$ on the Hamiltonian (V-1) will be studied. In particular we consider

$$H = J_1 + J_2 - J_1^2 - 3J_1J_2 + \alpha J_1J_2 \cos 2(\varphi_1 - \varphi_2) . \quad \text{(V-10)}$$

α is a parameter measuring the strength of the perturbation and will later be given specific values. As was mentioned in the previous paragraph, this system is in principle integrable since two integrals, $I = J_1 + J_2$ and H, are known. Thus we can find the level curves in the J_1-plane by setting $\varphi_2 = \frac{3}{2}\pi$, since Transformation (II-3) shows that $\varphi_2 = \frac{3}{2}\pi$ is equivalent to $x_2 = 0$ and $p_2 \geq 0$. The integral $I = J_1 + J_2$ can then be used to eliminate J_2 in Equation (V-10). The resulting equation for the level curves is

$$(3 + \alpha \cos 2\varphi_1)J_1^2 - J_1(5I + I \cos 2\varphi_1) + I + I^2 - E = 0, \quad \text{(V-11)}$$

We now intend to show that the level curves given by Equation (V-11) which includes the effects of the perturbation $\alpha J_1J_2 \cos 2(\varphi_1 - \varphi_2)$ are dramatically different from the concentric circle level curves of the unperturbed Hamiltonian at least in the neighborhood of the old $\Omega_1 = \Omega_2$ torus. This change occurs because the perturbations introduce new stable periodic solutions near the unperturbed $\Omega_1 = \Omega_2$ torus even when α is chosen as small as we please. The location of these new periodic solutions for Hamiltonian (V-10) could be determined by finding the extrema of Equation (V-11); however we choose to use the equations of motion

$$\dot{J}_1 = -\frac{\partial H}{\partial \varphi_1} = 2\alpha J_1J_2 \sin 2(\varphi_1 - \varphi_2) , \quad \text{(V-12a)}$$

$$\dot{\varphi}_1 = \frac{\partial H}{\partial J_1} = 1 - 2J_1 - 3J_2 + \alpha J_2 \cos 2(\varphi_1 - \varphi_2), \quad \text{(V-12b)}$$

$$\dot{J}_2 = -\frac{\partial H}{\partial \varphi_2} = -2\alpha J_1 J_2 \sin 2(\varphi_1 - \varphi_2), \qquad \text{(V-12c)}$$

$$\dot{\varphi}_2 = \frac{\partial H}{\partial J_2} = 1 - 3J_1 + 2J_2 + \alpha J_1 \cos 2(\varphi_1 - \varphi_2). \qquad \text{(V-12d)}$$

In looking for periodic solutions, we shall seek solutions of Equation (V-12) for which both J_1 and J_2 are constant. Hamiltonian (V-10) then implies implies that $(\varphi_1 - \varphi_2)$ is a constant unless J_1 or J_2 happens to be zero. The various cases will now be examined systematically.

One of the anomalous periodic solutions of Equation (V-12) is $J_1 = 0$ and $J_2 = I$. The angle equations could be easily integrated for this case but the resulting solutions would be of little interest. The level curve in the J_1-plane corresponding to this solution is, of course, a point located at the origin. Another periodic solution of this type is $J_2 = 0$ and $J_1 = I$. In the J_1 plane the level curve for this orbit is a circle centered on the origin. From Equation (V-10) the radius of this circle is found to be $\sqrt{1 - \sqrt{1 - 4E}}$. This circle is the bounding level curve in the J_1 plane. We note that both these periodic solutions are exceptional in that $(\varphi_1 - \varphi_2)$ need not be a constant.

The remaining periodic solutions are found by requiring that $\sin 2(\varphi_1 - \varphi_2)$ in Equations (V-12a,c) be zero for all times. There are four cases:

(a) $\varphi_1 - \varphi_2 = 0$

For this to remain true for all times, J_1 and J_2 must be such that $\dot{\varphi}_1 = \dot{\varphi}_2$. From Equation (V-12) the required relationship between the J's is

$$J_1 = \left(\frac{5-\alpha}{1-\alpha}\right) J_2 \ . \qquad \text{(V-13)}$$

The level curve representing this solution is a point on the positive p_1 axis. The position of this point is a function of energy and can be found using Equations (V-13) and (V-10).

(b) $\varphi_1 - \varphi_2 = \pi$

The relationship between the action variables is found to be the same as in Equation (V-13). The representative level curve is a point on the negative p_1 axis.

(c) $\varphi_1 - \varphi_2 = \pi/2$

Again using Equation (V-12) we find that

$$J_1 = \left(\frac{5+\alpha}{1+\alpha}\right) J_2 \ . \qquad \text{(V-14)}$$

The level curve is a point on the positive x_1 axis.

(d) $\varphi_1 - \varphi_2 = 3\pi/2$

The relationship between J_1 and J_2 is the same as Equation (V-14) and the level curve is a point on the negative x_1 axis.

In each of the four cases above, once the values of J_1 and J_2 are found for a given energy, the constant angular frequencies, and hence the angles, φ_1 and φ_2, can be calculated from Equations (V-12b) and (V-12d).

These periodic solutions represent extrema of Equation (V-11), and, by using the usual derivative tests, it is straightforward to show that the two periodic solutions on the x_1 axis are stable, while the two on the p_1 axis are unstable. A typical set of level curves for this

Hamiltonian with E = 0.10 and α = 0.95 is shown in Figure 10. These curves were obtained by solving Equation (V-11) for J_1 as a function of φ_1 and then converting to Cartesian coordinates. Changes in E and α produce distortions and shifts in location for the curves, but the basic pattern remains essentially the same.

Note that in all of the four cases listed above, the relationship between J_1 and J_2 is, for small alpha, close to the equation $J_1 = 5J_2$ used in obtaining Equation (V-4) which specify the torus with equal frequencies in the unperturbed problem. The addition of the 2-2 resonance term has radically altered the tori in the neighborhood of the unperturbed $\Omega_1 = \Omega_2$ torus, creating the new stable and unstable periodic solutions shown in Figure 10. However, away from this neighborhood the tori are only slightly deformed. The two crescent-shaped regions in Figure 10 bounded by a separatrix are reminiscent of the island chains in the Henon and Heiles work. However this is not a chain of two islands since the central invariant points represent periodic solutions for which all four variables have the same period. To obtain an island chain, periodic solutions must exist for which the period of the x_1 motion is different from that of the x_2 motion.

As an example of an island chain, we shall consider the effect of adding a resonance term proportional to cos $(3\varphi_1 - 2\varphi_2)$ to the unperturbed Hamiltonian (V-1). Using β as the "strength" parameter, the Hamiltonian becomes

$$H = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2 + \beta J_1^{3/2} J_2 \cos(3\varphi_1 - 2\varphi_2) \qquad \text{(V-15)}$$

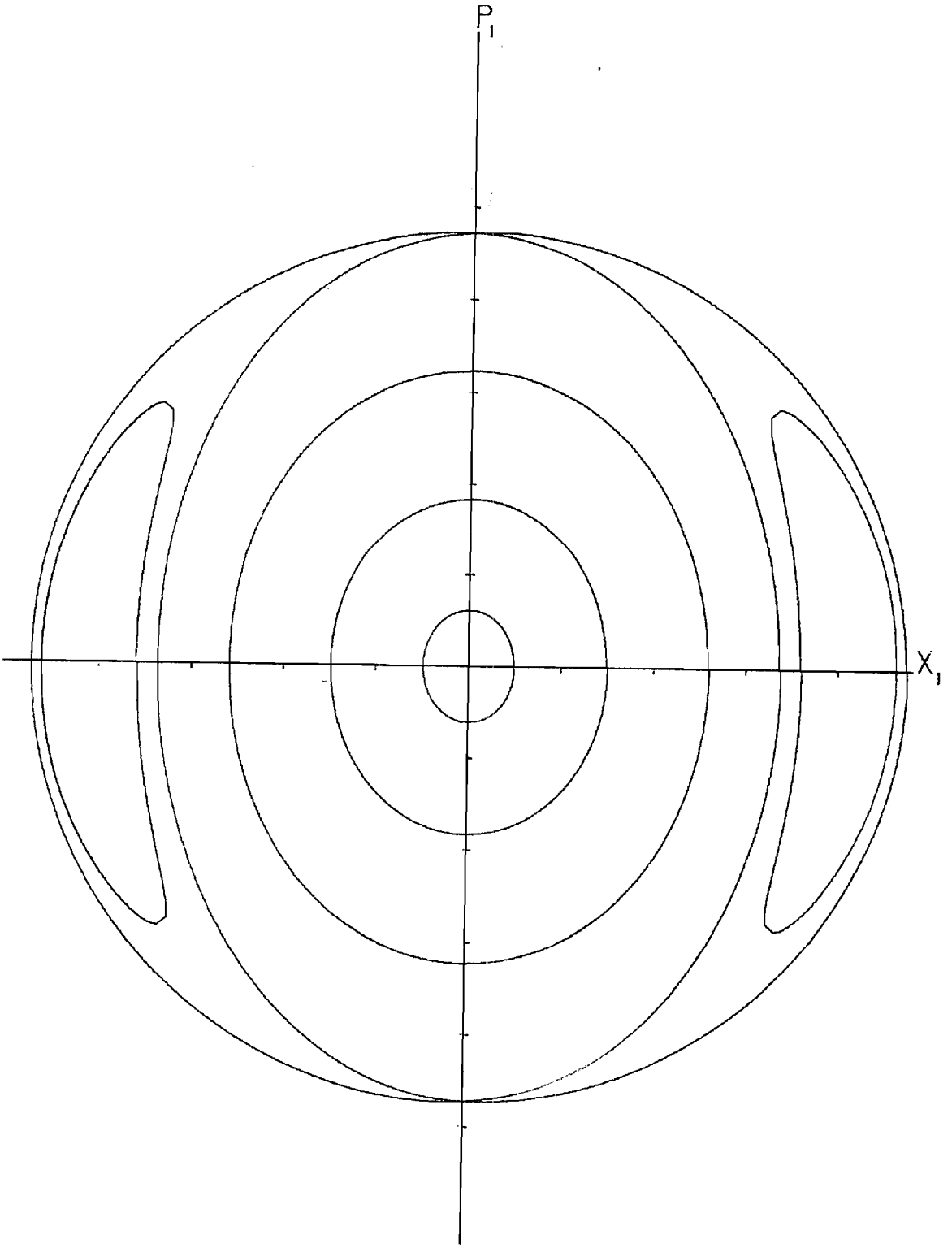

Figure 10. Level Curves for the Pure 2-2 Resonance.
Energy E = 0.10, α = 0.95.

where $\cos(3\varphi_1 - 2\varphi_2)$ has been multiplied by $J_1^{3/2} J_2$, the lowest power of the action variables which will keep the Hamiltonian analytic at the origin.

The dynamical system described by Equation (V-15) is integrable. The additional integral is

$$\bar{I} = 2J_1 + 3J_2 \qquad \text{(V-16)}$$

as may be verified by calculating the time derivative of $\bar{I}$ and then using the equations of motion. In order to find level curves in the J_1-plane, we again set $\varphi_2 = \frac{3\pi}{2}$ in Equation (V-15) and use Equation (V-16) to eliminate J_2. The level curves are then given implicitly by

$$E = \frac{\bar{I}}{3} + \frac{\bar{I}^2}{9} + J_1\left(\frac{1}{3} - \frac{13}{9}\bar{I}\right) + \frac{13}{9} J_1^2 - \frac{\beta}{3}\left(\bar{I} J_1^{3/2} - 2J_1^{5/2}\right)\cos 3\varphi_1 \qquad \text{(V-17)}$$

Here we expect the unperturbed tori near $3\Omega_1 = 2\Omega_2$ to be seriously altered due to the appearance of new periodic solutions. These new periodic solutions are found using the equations of motion

$$\dot{J}_1 = 3\beta J_1^{3/2} J_2 \sin(3\varphi_1 - 2\varphi_2), \qquad \text{(V-18a)}$$

$$\dot{\varphi}_1 = 1 - 2J_1 - 3J_2 + 3\frac{\beta}{2} J_1^{1/2} J_2 \cos(3\varphi_1 - 2\varphi_2), \qquad \text{(V-18b)}$$

$$\dot{J}_2 = -2\beta J_1^{3/2} J_2 \sin(3\varphi_1 - 2\varphi_2), \qquad \text{(V-18c)}$$

$$\dot{\varphi}_2 = 1 - 3J_1 + 2J_2 + \beta J_1^{3/2} \cos(3\varphi_1 - 2\varphi_2). \qquad \text{(V-18d)}$$

Once again there are the anomalous periodic solutions, $J_1 = 0$, $J_2 = \bar{I}/3$,

and $J_1 = \bar{I}/2$, $J_2 = 0$, representing the origin and the bounding level curve respectively. The other periodic solutions are obtained from the condition $\sin(3\varphi_1 - 2\varphi_2) = 0$. There are only two cases;

(a) $3\varphi_1 - 2\varphi_2 = 0, 2\pi, 4\pi, \ldots$

Values of J_1 and J_2 must be determined such that $\sin(3\varphi_1 - 2\varphi_2) = 0$ for all times. Using Equations (V-18b) and (V-18d) and requiring that $3\dot{\varphi}_1 = 2\dot{\varphi}_2$ gives

$$J_2 = \frac{1 - 2\beta J_1^{3/2}}{13 - \frac{9}{2}\beta J_1^{1/2}} . \qquad \text{(V-19)}$$

Equation (V-19) can be used in Equation (V-15) to find values of J_1 and J_2 for this periodic solution. The resulting equation is of seventh degree and must be solved numerically. The level curve for this periodic solution in the J_1-plane has J = constant. We then set φ_2 equal to $3\pi/2$. For $(3\varphi_1 - 2\varphi_2) = 0$, there is an isolated point on the level curve at $\varphi_1 = \pi$. $(3\varphi_1 - 2\varphi_2) = 2\pi$ gives a point at $\varphi_1 = 5\pi/3$ and $(3\varphi_1 - 2\varphi_2) = 4\pi$ gives a point at $\varphi_1 = \pi/3$. Other multiples of 2π give a repetition of this pattern. Thus this periodic solution intersects the J_1-plane at three isolated points.

(b) $3\varphi_1 - 2\varphi_2 = \pi, 3\pi, 5\pi, \ldots$

Computing $3\dot{\varphi}_1 = 2\dot{\varphi}_2$ from Equations (V-13b) and (V-18d) gives J_2 as a function of J_1 for this periodic orbit. We find that

$$J_2 = \frac{1 + 2\beta J_1^{3/2}}{13 + \frac{9}{2}\beta J_1^{1/2}} . \qquad \text{(V-20)}$$

Equation (V-20) together with Equation (V-15) determine J_1 and J_2 and hence the periodic solution. Again setting $\varphi_2 = \frac{3\pi}{2}$, we find that the level curve in the J_1-plane consists of three points with $\varphi_1 = 0$, $2\pi/3$, and $4\pi/3$.

The branch problem of selecting the correct J_1-root here is not as easily resolved as with Hamiltonian (V-1) although in principle the same criterion could be used. For energies greater than the minimum energy determined from Equation (V-5) for the existence of the $3\Omega_1 = 2\Omega_2$ torus in the unperturbed problem, Equation (V-15) yields only three non-negative roots for J_1. It is easy to choose the proper root on the basis of magnitude alone. For energy less than 14/169, there are no allowed roots.

An examination of the extrema of Equation (V-17) shows that the periodic solution determined by Equation (V-19) is unstable while the one determined by Equation (V-20) is stable. A set of level curves for $E = 0.136$ and $\beta = 0.25$ is shown in Figure 11. These level curves were calculated from Equation (V-17), which is of fifth degree in $\sqrt{J_1}$, using a modified Lehmer's method (Delves and Lyness 1967). The most prominent feature is the chain of three islands and the associated separatrix about the origin. In the J_2-plane there is a chain of two islands. At higher energies the general pattern of level curves is unchanged, while below $E = 14/169$ the level curves are slightly distorted circles centered on the origin. Finally we note that, as in the previous example, the region where the tori are radically changed is in the

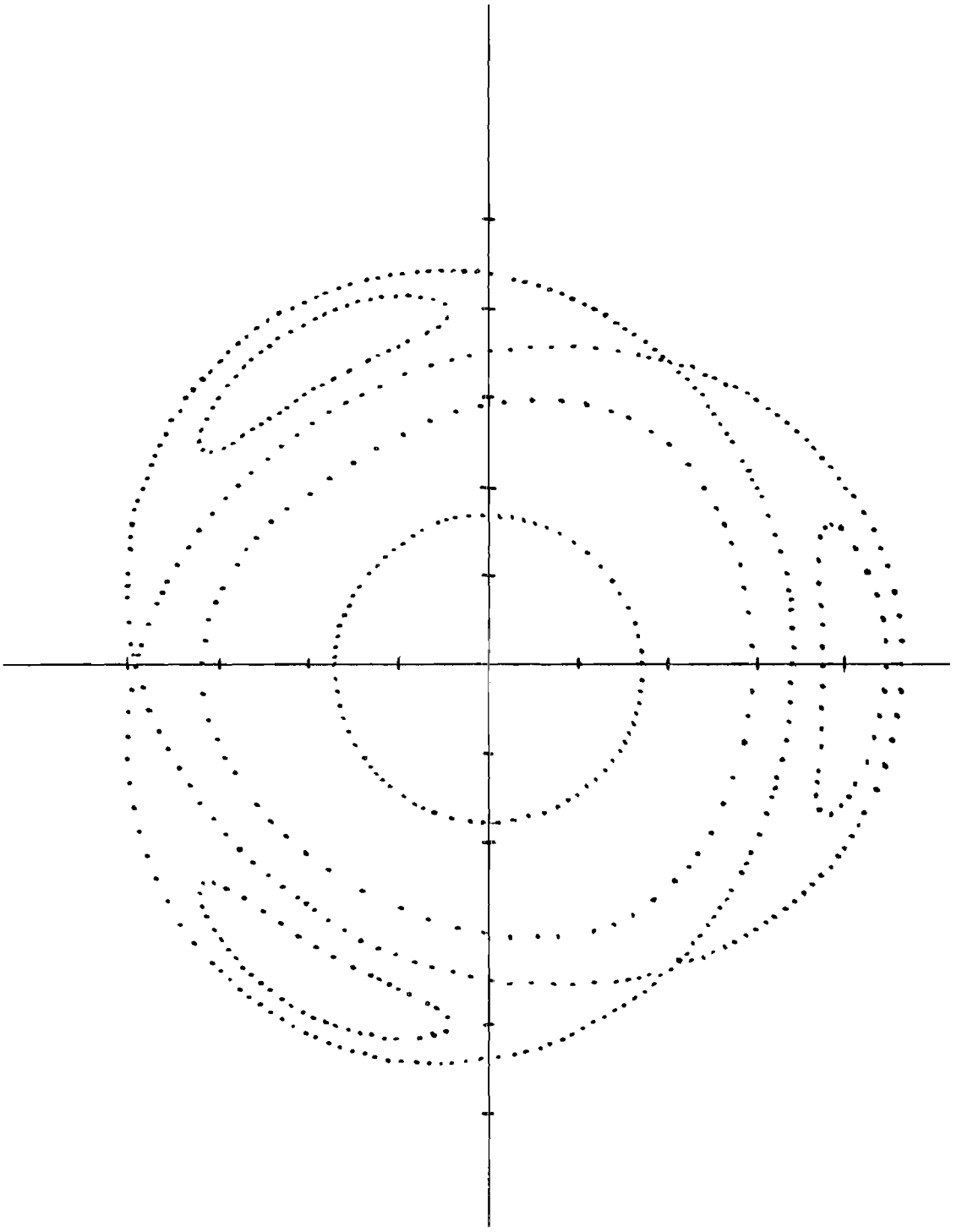

Figure 11. Level Curves for the Pure 3-2 Resonance.
Energy E = 0.136 and β = 0.25.

neighborhood of the unperturbed $3\Omega_1 = 2\Omega_2$ torus. In fact for small values of β, Equations (V-19) and (V-20) are almost the same as the condition $J_2 = 1/13$ for finding the unperturbed $3\Omega_1 = 2\Omega_2$ torus.

The last example to be considered is a perturbation that will destroy the $2\Omega_1 = 3\Omega_2$ torus. For this case we choose the Hamiltonian

$$H = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2 + \beta J_1 J_2^{3/2} \cos(2\varphi_1 - 3\varphi_2). \quad \text{(V-21)}$$

The method of analysis used in the last two examples is also applicable here without essential change. We shall merely state the results. There is an additional integral given by

$$A = 3J_1 + 2J_2 . \quad \text{(V-22)}$$

The level curves in the J_2-plane, for example, are found from

$$E = \frac{A}{3} - \frac{A^2}{9} + (\frac{1}{3} - \frac{5}{9} A)J_2 + \frac{23}{9} J_2^2 + \beta(\frac{2}{3} J_2^{5/2} - \frac{A}{3} J_2^{3/2}) \cos 3\varphi_2 . \quad \text{(V-23)}$$

The nontrivial periodic solutions are determined by

$$2\varphi_1 - 3\varphi_2 = 0, 2\pi, 4\pi, \ldots . \quad \text{(V-24a)}$$

$$J_1 = \frac{1 - 2\beta J_2^{3/2} + 12 J_2}{5 - \frac{9}{2}\beta J_2^{1/2}} \quad \text{(V-24b)}$$

and

$$2\varphi_1 - 3\varphi_2 = \pi, 3\pi, 5\pi, \ldots \quad \text{(V-25a)}$$

$$J_1 = \frac{1 + 2\beta J_2^{3/2} + 12 J_2}{5 + \frac{9}{2}\beta\, J_2^{1/2}} \qquad \text{(V-25b)}$$

The periodic solutions obtained from Equation (V-24) are unstable, while those obtained from Equation (V-25) are found to be stable. The level curves are similar to those of the previous example except that the chain of three islands is in the J_2-plane.

The results from these examples are typical of the behavior for Hamiltonians having the form of Equation (V-9). The unperturbed torus leading to circular level curves with $m\Omega_1 = n\Omega_2$ is found from

$$m \frac{\partial H_o}{\partial J_1} = n \frac{\partial H_o}{\partial J_2} \qquad \text{(V-26)}$$

In the perturbed problem new periodic solutions are found by solving $m\dot{\phi}_1 = n\dot{\phi}_2$ or

$$m\left(\frac{\partial H_o}{\partial J_1} + \frac{\partial V}{\partial J_1}\right) = n\left(\frac{\partial H_o}{\partial J_2} + \frac{\partial V}{\partial J_2}\right). \qquad \text{(V-27)}$$

For small perturbations V we see that these new periodic solutions are "close" to the $m\Omega_1 = n\Omega_2$ torus found from Equation (V-26). The width and precise location of the "destroyed" region may be determined from analytic level curve expressions like Equations (V-11), (V-17), and (V-23); they are dependent upon the energy and upon the perturbation strength, increasing with increasing energy and perturbation strength.

Having now established that integrable Hamiltonians can give rise

to island chains under the influence of isolated resonances, one quite naturally wonders what would happen if two or more such resonances acted simultaneously. Suppose for example that the 2-2 and 3-2 resonances of Hamiltonians (V-10) and (V-15) were simultaneously applied with α and β both small. We then might expect the 2-2 resonance to establish crescent-shaped level curves in the vicinity of the unperturbed $\Omega_1 = \Omega_2$ level curve and the 3-2 resonance to generate a chain of three islands in the neighborhood of the unperturbed $3\Omega_1 = 2\Omega_2$ level curve with the remaining level curves being slightly distorted circles centered on the origin. Moreover, one would expect some technique could be devised which could predict the level curves within each region since only one resonance at most is dominant anywhere. However, if the energy or α and β were increased sufficiently, the two isolated resonances could overlap and simultaneously influence level curves in the same region. Might not seemingly ergodic trajectories arise in this region due to the appearance of a profusion of stable and unstable periodic orbits? In the following chapter we investigate this question and in the final chapters, we attempt to relate our discoveries to the Henon-Heiles system.

CHAPTER VI

MULTIPLY RESONANT DYNAMICAL SYSTEMS

The typical Hamiltonian when expressed in action-angle variables will generally have, in addition to the terms involving only the action, nonresonant angle dependent terms and one or more distinct resonance terms. Since the nonresonant angle dependent terms can be eliminated using a Birkhoff-type transformation, they will not be considered further in this study. In this chapter we shall consider Hamiltonians with more than one distinct resonance term and we shall say that such Hamiltonians are multiply resonant. In particular we shall study two, doubly-resonant, model Hamiltonians using the results from Chapter V to make predictions about the behavior of these model systems. Finally, since multiply resonant Hamiltonians do not have any obvious third isolating integrals, our predictions will have to be tested by numerically integrating the equations of motion.

For the sake of concreteness we shall consider the effects of adding two resonance terms to Hamiltonian (V-1). As previously mentioned, it is convenient to specify the type of resonance term as an "m-n resonance" meaning that it is a function of the action variables, J_1 and J_2, multiplied by $\cos(m\varphi_1 - n\varphi_2)$. It will also be convenient in the following to specify tori or level curves simply by listing their frequency ratios, an m-n torus or level curve being one for which the frequency ratio is m/n. Since the 2-2 and 2-3 resonances were studied in detail in Chapter

V, we shall add these terms to Hamiltonian (V-1) giving

$$H = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2 + \alpha J_1J_2 \cos 2(\phi_1 - \phi_2) + \beta J_1J_2^{3/2} \cos(2\phi_1 - 3\phi_2) \qquad \text{(VI-1)}$$

Note that if $\alpha = 0$ this is Hamiltonian (V-21) and if $\beta = 0$ this is Hamiltonian (V-10). Let us recall from Figure 10 in Chapter V that the 2-2 resonance term acting alone on Hamiltonian (V-1) only slightly distorts the unperturbed circular level curves except in the neighborhood of the unperturbed 2-2 level curve. Near the 2-2 (or equivalently the 1-1) torus, the 2-2 resonance produced a radical change in the unperturbed system motion by introducing new stable and unstable periodic solutions and a separatrix supporting asymptotic motion. The action of the 2-3 perturbation on Hamiltonian (V-1) is similar. In this latter case the region of maximum change in the level curves is in the neighborhood of the unperturbed 2-3 level curve.

Since the two resonant terms in Hamiltonian (VI-1) when acting alone serve to grossly distort the circular unperturbed level curves only in relatively small disjoint regions of the J_2-plane, one might expect this situation to be essentially unchanged when the energy is very small and α and β are both nonzero but small. Indeed this expectation can be rigorously justified using the Kolmogorov scheme mentioned near Equation (IV-12). In particular consider Hamiltonian (VI-1) for initial conditions near the unperturbed 2-2 tori. Using the Kolmogorov scheme, one may for most of the initial conditions transform Hamiltonian (VI-1) to the form

$$H = \mathcal{J}_1 + \mathcal{J}_2 - \mathcal{J}_1^2 - 3\mathcal{J}_1\mathcal{J}_2 + \mathcal{J}_2^2 + \alpha\mathcal{J}_1\mathcal{J}_2\cos 2(\theta_1 - \theta_2) + \beta f(\mathcal{J}_1, \mathcal{J}_2), \tag{VI-2}$$

where f is some function of the action variables and where the $\mathcal{J}_i$ differ only slightly from the J_i. Thus for most initial conditions in the neighborhood of the unperturbed 2-2 torus, the 2-3 resonance serves only to shift slightly the frequencies. As a consequence, the level curves for Hamiltonians (VI-2) or (VI-1) in the neighborhood of the unperturbed 2-2 torus should be almost the same as the $\beta = 0$ tori of Figure 10. This becomes especially evident if one notices that Hamiltonians (VI-2) and (V-10) are essentially the same. A similar argument shows that the 2-2 resonance only serves to shift slightly the frequencies in the neighborhood of the unperturbed 2-3 torus and hence the level curves there are almost the same as the $\alpha = 0$ level curves. Kolmogorov's theory then leads to the following picture for small energies or for α and β small but nonzero. Near the unperturbed 2-2 tori the level curves for Hamiltonian (VI-1) should include a separatrix bearing two unstable periodic solutions, surrounding two stable periodic solutions as in Figure 10. The positions of these periodic solutions should be close to the positions of the periodic solutions for Hamiltonian (V-10) calculated from Equations (V-13) and (V-14). In addition, if the energy is greater than 0.16, there should be a chain of three islands (with a separatrix) at approximately the positions predicted by Equations (V-24) and (V-25). The remaining level curves should be slightly distorted concentric circles centered on the origin. Before turning to computer verification of these points, let us investigate the situation for which

the energy or α and β are not small. In particular we wish to know if the 2-2 and 2-3 resonances can ever strongly affect the same unperturbed level curve for sufficiently large energies or large α and β. In order to assay this possibility, we determine the size and position of the regions of influence for the 2-2 and 2-3 resonances when each acts alone. For this purpose we shall use the equations developed in Chapter V.

A region in our four dimensional phase space which is enclosed by a separatrix (which is a two dimensional surface) produced by an m-n resonance will be called an "m-n resonance zone." For example, the level curve representation of the 2-2 resonance zone for Hamiltonian (V-10) is the double crescent region of Figure 10. A measure of the "width" of such a resonance zone can be obtained by determining the width of one of the crescents. In particular, we determine the distance between the two points of intersection of the separatrix level curve with the positive x_2 axis (for level curves in the J_2-plane). The width of the 2-2 and 2-3 resonance zones were determined as functions of α and E_2 and β and E_2 respectively using Equations (V-11,13,23,24). It was found in both cases that the width is a monotonically increasing function of energy and perturbation strength. Next, we determined the position of these resonance zones in phase space. We then calculated the energy for which the resonance zones overlap for fixed α and β. The method used is to establish the positions of the various separatrices using level curves. For example in Figure 10, the inner intersection of the 2-2 separatrix with the x_2 axis represents a minimum distance from the origin to the 2-2 resonance zone. This distance is zero for zero

energy and increases monotonically with energy. Similarly, the outer intersection of the 2-3 separatrix with the positive x_2 axis represents a maximum distance from the origin to the 2-3 resonance zone. Since the 2-3 resonance zone lies closer to the origin than does the 2-2 zone (see Figure 14), overlap first occurs when the outer separatrix of the 2-3 zone touches the inner separatrix of the 2-2 zone. For energy less than 0.16, the 2-3 zone does not exist. Thereafter, the outer edge of this zone starts at the origin for $E = 0.16$ and moves outward monotonically as the energy increases. The distances from the origin to the outer edge of the 2-3 zone and to the inner edge of the 2-2 zone were calculated numerically as functions of the energy for $\alpha = 0.02$ and $\beta = 0.02$. The results of this computation are displayed graphically in Figure 12. It is clear that for energies less than 0.16 the two zones are certainly widely separated as one of them is nonexistent. At $E = 0.2095$ we see from Figure 12 that the regions begin to overlap. We thus anticipate that our model Hamiltonian (VI-1) may exhibit irregular behavior similar to that of Henon and Heiles system provided that $E \geq 0.2095$.

In order to test the validity of the above arguments, the level curves for Hamiltonian (VI-1) with $\alpha = \beta = 0.02$ were found by numerically integrating the equations of motion. Since the chain of three islands does not appear until $E = 0.16$, as a first test the energy was fixed at 0.0561. The level curves for this energy are shown in Figure 13.

For energies in this range, the two perturbations in Hamiltonian (VI-1) should act almost independently. As a check, the location for the stable periodic solutions for Hamiltonian (V-10) was calculated from Equation (V-13) yielding $x_2 \simeq .142$. From a graph similar to Figure 13, we

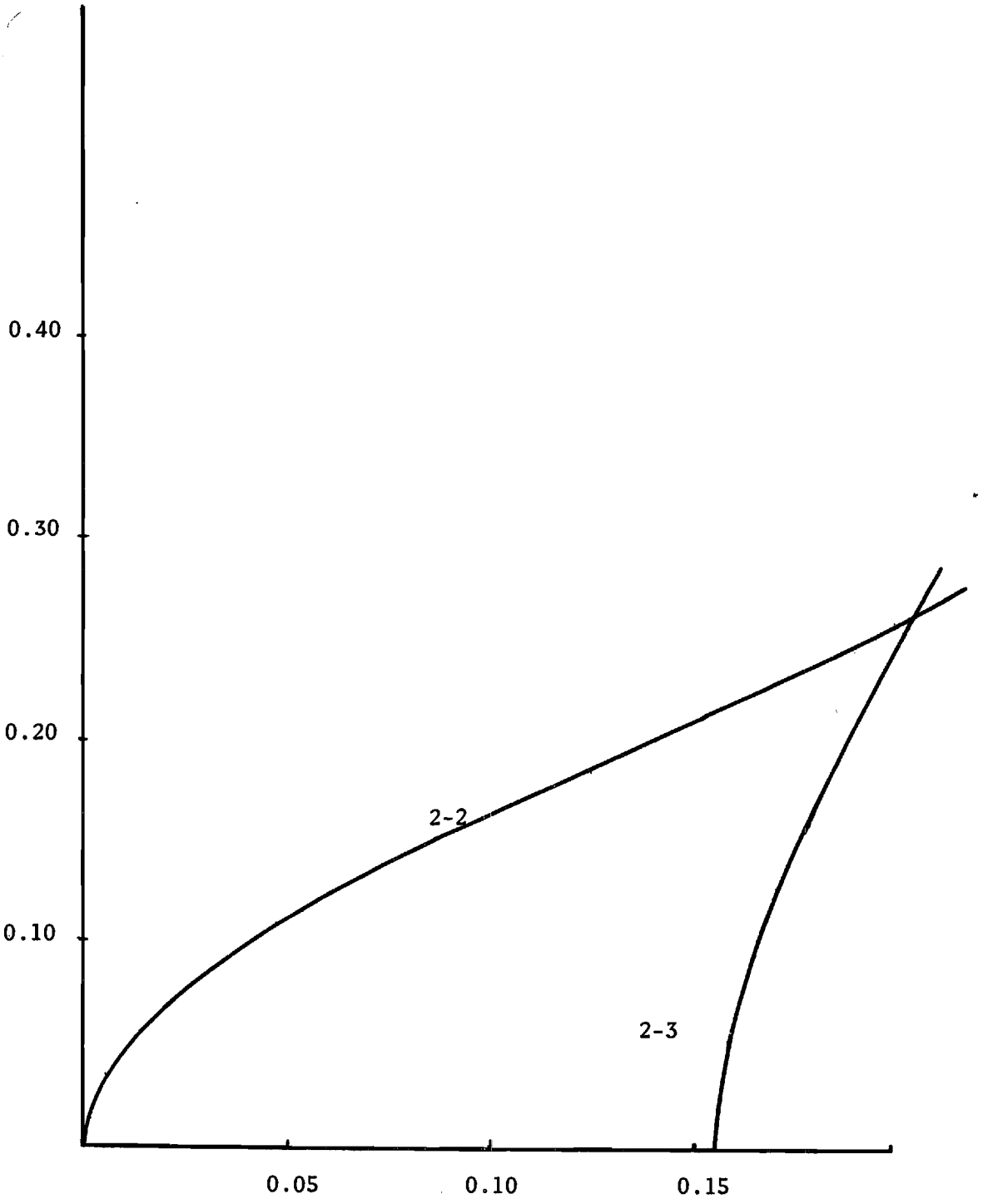

Figure 12. Position of the 2-2 and 2-3 Separatrices as a Function of Energy.

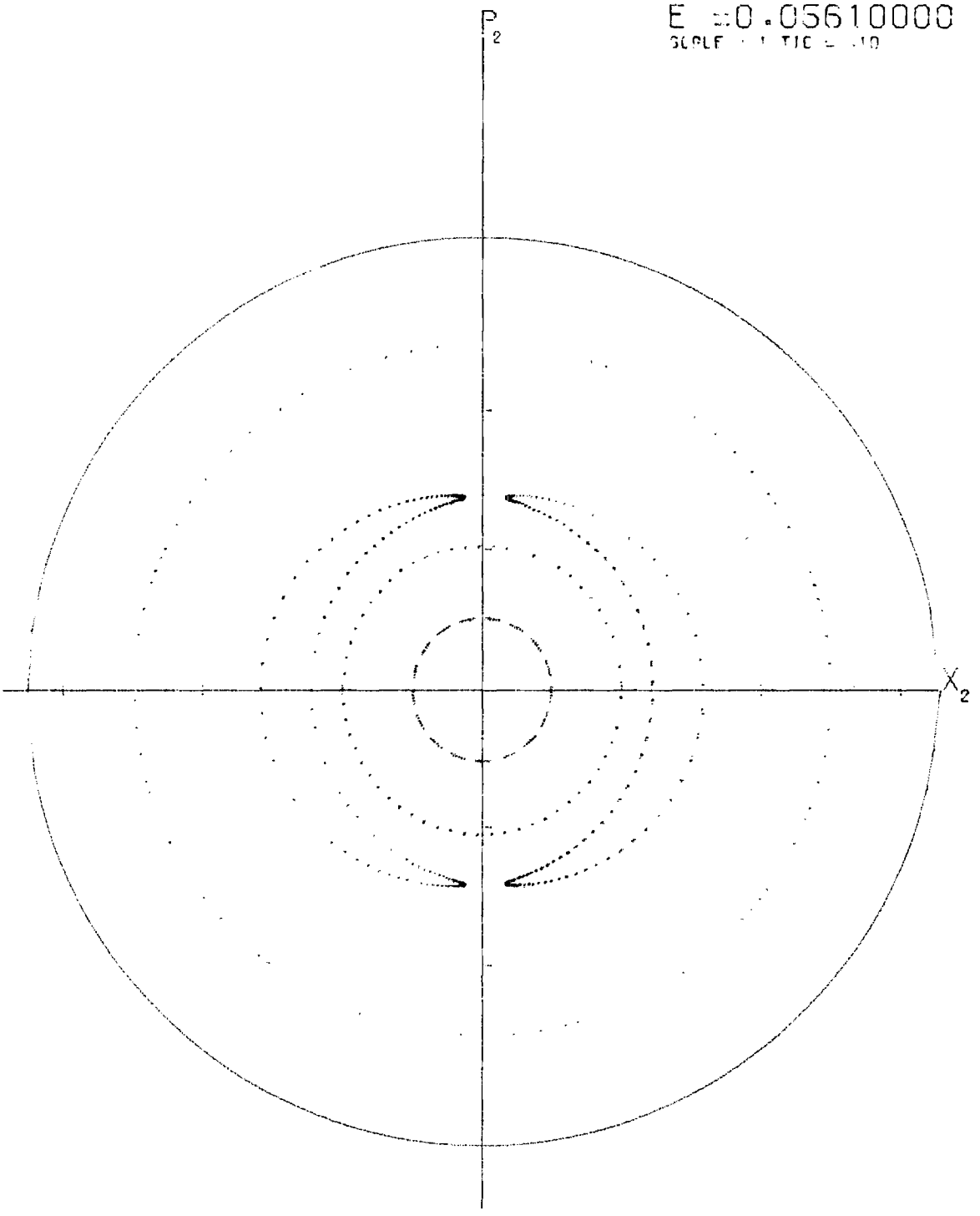

Figure 13. Level Curves for the First Model Hamiltonian.

found that the location of the periodic solution with both perturbations was $x_2 \cong 0.142$. If $\beta = 0$, then $I = J_1 + J_2$ is an integral of Hamiltonian (VI-1). On the other hand, if $\alpha = 0$, then $\bar{I} = 3J_1 + 2J$ is an integral. Of course if $\alpha \neq 0$ and $\beta \neq 0$, then neither I nor $\bar{I}$ are integrals; however, if α and β are small, we might expect I and $\bar{I}$ to be approximately conserved. I and $\bar{I}$ were calculated along with the energy for the orbits used in finding the level curves. The energy behaved as in the Henon and Heiles example of Chapter II, being conserved to six figures. I was found to be constant at this energy to at least four figures and generally to around six figures. $\bar{I}$ was not so well-behaved and was constant only through two figures. The energy in the two perturbing terms always remained small- usually around .03 per cent for the 2-2 resonance and .002 per cent for the 2-3 resonance.

For energies greater than 0.16 the 2-3 resonance acting alone in Hamiltonian (VI-1) produces a chain of three islands in the J_2-plane. It is of interest to see whether or not this behavior persists for Hamiltonian (VI-1). As a check then, the level curves for Hamiltonian (VI-1) were calculated for $E = 0.18$ with $\alpha = \beta = .02$. The results are shown in Figure 14. Again the stable periodic solutions are in the location and orientation by integrable Hamiltonians for which $\alpha = 0$ or $\beta = 0$. Moreover, the calculated points in Figure 14 all seem to lie on closed level curves. The total energy was conserved again to within six figures while the values of $I = J_1 + J_2$ and $\bar{I} = 3J_1 + 2J_2$ were constant to around three figures.

In the tests discussed above, prediction based on treating the two perturbing terms as though they acted separately in Hamiltonian

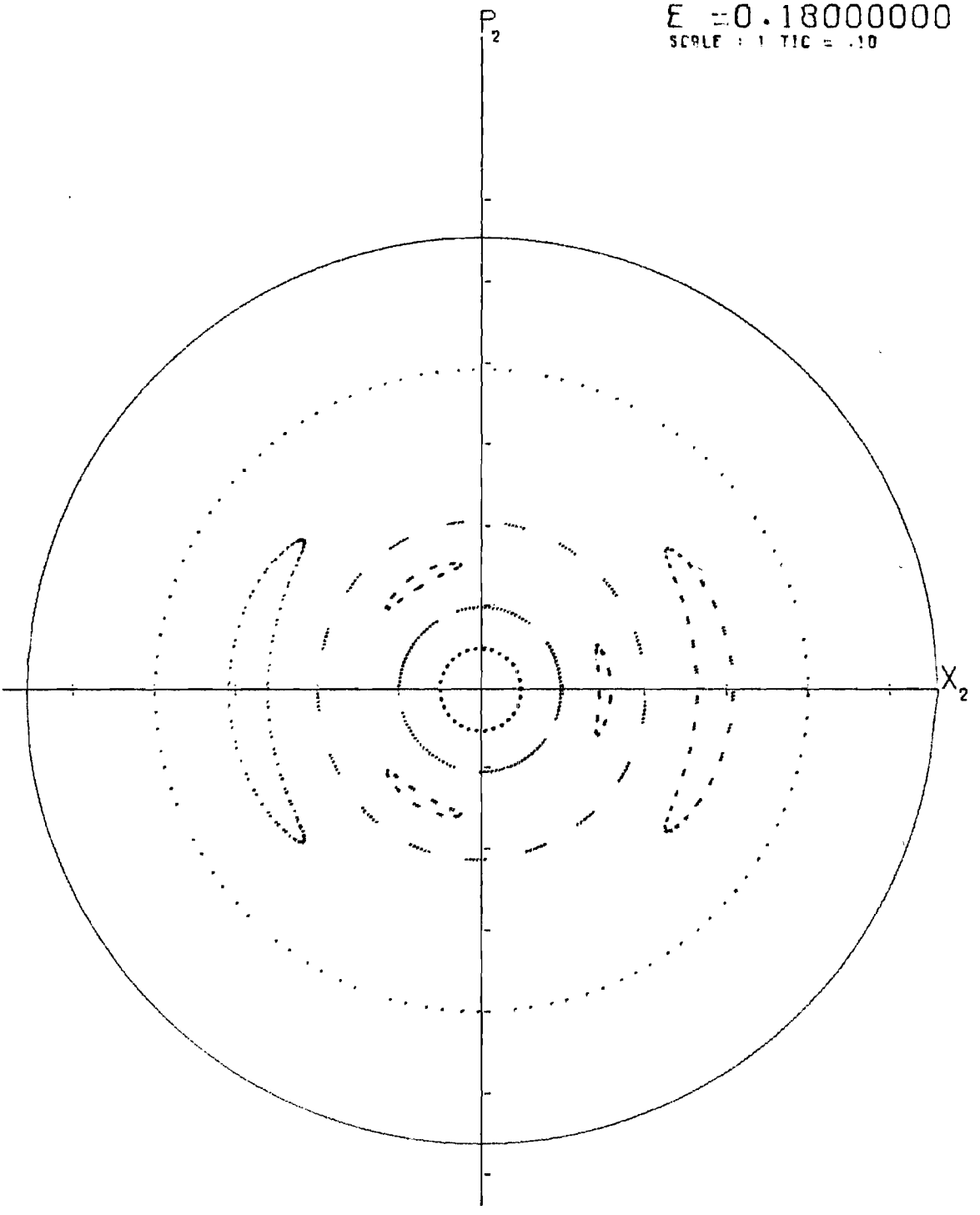

Figure 14. Level Curves for the First Model Hamiltonian.

(VI-1) has been verified for the energies studied. However, when the two resonances overlap or nearly overlap, one no longer expects the two perturbing terms to act independently. Using Hamiltonians (V-10) and (V-21), we have that at energy E = 0.2095 the 2-2 and 2-3 separatrices for Hamiltonian (VI-1) should first come into contact. Since it is not clear what behavior to expect for Hamiltonian (VI-1) at this energy, it is instructive to calculate the level curves. In particular we would like to test the hypothesis that this energy marks the start of the breakdown of the level curves. The results are shown in Figure 15. Note that the level curves in the region around what would have been the two separatrices have indeed broken down, giving a sequence of seemingly ergodic points similar to those obtained for the Henon and Heiles system shown in Figure 4. However, the chain of three islands still remains; only its associated separatrix is gone. Likewise, there are closed level curves inside the destroyed 2-2 separatrix. Outside the regions of influence of the 2-2 and 2-3 resonances, the level curves continue to exist and are almost circular. During the calculation of these level curves by integration of the equations of motion, the total energy, as before, remained constant to six figures. The variations of $J_1 + J_2$ and $3J_1 + 2J_2$ are, as one might expect, much greater than before, each being constant to only one figure. The energy in the perturbation remained small, around 0.1 per cent of the total energy.

We see then that in regions where the resonance zones overlap (as predicted using integrable systems), the separatrices and some of the level curves are destroyed or else replaced by some kind of very complicated structure. In particular, a band of level curves near the integrable

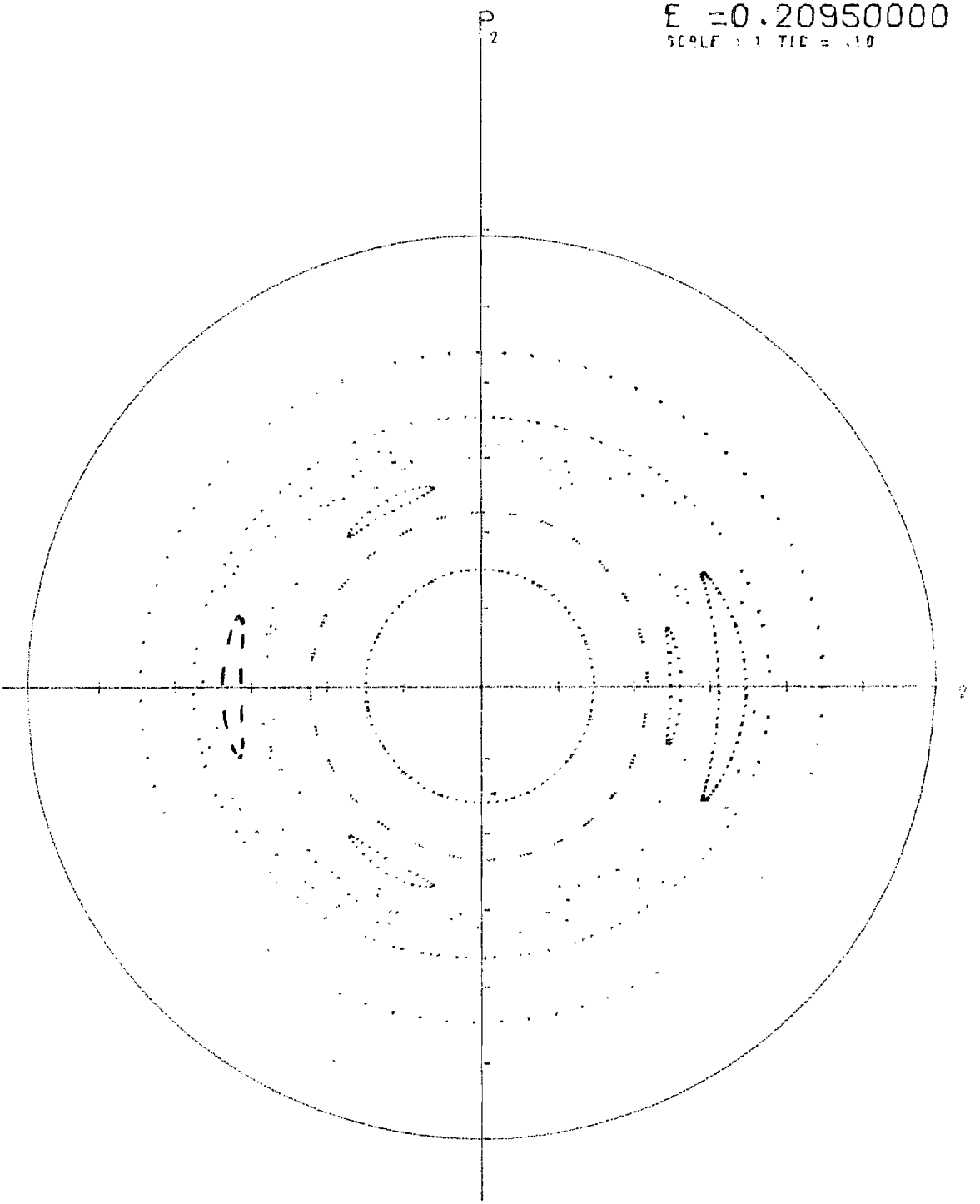

Figure 15. Level Curves for the First Model Hamiltonian.

system separatrices have been replaced by an ergodic-type region. Thus the very simple procedure of finding the energy and location of the separatrix overlap for the two "uncoupled" Hamiltonians (V-10) and (V-21) enables us to predict approximately the energy and region of breakdown for the level curves of Hamiltonian (VI-1).

In order to provide a more careful check on the accuracy of this procedure, it is necessary to examine the level curves at an energy slightly below 0.2095 to see if perhaps breakdown occurs before the predicted 2-2, 2-3 overlap. In doing so, it is wise to be aware of the full complexity of this problem. As was previously mentioned above Equation (VI-2), the two resonance terms do not act independently in Hamiltonian (VI-1) for all initial conditions. There is a very complicated hierarchy of effects which we now proceed to discuss. First we introduce some useful terminology. Any resonance term appearing explicitly in a Hamiltonian will be termed a "primary resonance." The large scale effects of primary resonances acting on angle independent Hamiltonians have been discussed in detail above. As we shall now indicate, the primary resonances may interact with each other producing higher order "secondary" resonances. The effects of these secondary resonances are generally small and hard to detect on a computer. However for sufficiently large energies, they can produce noticeable effects on the level curve diagrams.

We now examine the origin of these secondary resonance terms. Hamiltonian (VI-1) may be regarded as Hamiltonian (V-10) with a 2-3 resonant perturbing term. If we neglect the 2-3 perturbing term, it is possible to introduce new action-angle variables, $\mathcal{J}_i$, θ_i, by a canonical transformation T in such a way that Hamiltonian (V-10) becomes a function

of the action alone (Whittaker 1961). Denoting this transformed Hamiltonian by H_1, we have

$$H_1 = H_1(\mathcal{J}_1, \mathcal{J}_2) \ . \qquad \text{(VI-3)}$$

The level curves of Hamiltonian (VI-3) are circles centered on the origin. The details of what happens to the separatrix and the stable periodic solutions under the transformation T need not concern us here. The new constant frequencies, as given by the equations of motion arising from Hamiltonian (VI-3), are

$$\bar{\omega}_1 = \frac{\partial H_1}{\partial \mathcal{J}_1} \ , \qquad \text{(VI-4a)}$$

$$\bar{\omega}_2 = \frac{\partial H_1}{\partial \mathcal{J}_2} \ . \qquad \text{(VI-4b)}$$

Under the transformation T the 2-3 resonance in Hamiltonian (VI-1) becomes some function of the new coordinates, $V(\mathcal{J}_1, \mathcal{J}_2, \theta_1, \theta_2)$. Hamiltonian (VI-1) in the new coordinates is

$$H = H_1(\mathcal{J}_1, \mathcal{J}_2) + \beta V(\mathcal{J}_1, \mathcal{J}_2, \theta_1, \theta_2) \ . \qquad \text{(VI-5)}$$

If $V(\mathcal{J}_1, \mathcal{J}_2, \theta_1, \theta_2)$ is expanded in a Fourier series in the new angle variables, then in general a number of new resonances will appear. These new resonances which are in a sense latent in Hamiltonian (VI-1), are what we have designated as secondary resonances. If the resonance zones of these secondary resonances are spatially separated, then we may treat them as we did the primary resonances. For example, the

secondary resonance proportional to $\cos(m\theta_1 - n\theta_2)$ will break down the tori in some neighborhood of the unperturbed m-n torus of Hamiltonian (VI-3), creating a new chain of islands. Moreover since the expansion of the perturbation V in a Fourier series will in general include an infinity of terms, there will be an infinite number of secondary resonances, each creating an island chain. However, the functions of the action and the powers of β occurring in these secondary resonances are very small for small amplitudes of motion and small β, and our calculations, as well as those of Henon and Heiles, indicate that the widths of the island chains become progressively smaller with increasing order. Hence it is possible to have the resonance zones separated by preserved, slightly distorted circular tori centered on the origin. As a consequence, at low energies one anticipates following Kolmogorov that, although the actual level curves are very complicated, most of the tori for Hamiltonian (VI-3) are preserved and the system motion is definitely non-ergodic. At higher energies we would expect the width of the secondary resonance zones to increase, resulting in overlap and giving rise on a smaller scale to the behavior observed when the two primary resonances overlapped.

As a final complicating factor, we note that some of these resonances may create island chains in the regions of $\mathcal{J}_i, \theta_i$ space which transform back into the interior of the chain of three islands in J_i, φ_i space. It is thus possible (Jeffreys 1966) to have island chains within island chains. Clearly this process can be repeated any number of times. The angle independent terms in Hamiltonian (VI-5) can be grouped together with one of the resonance terms. Let H_2 denote this Hamiltonian. This integrable system can then be transformed by introducing new action-angle

variables in such a way that H_2 becomes a function of the new action variables alone. New latent resonances are perhaps exposed by this transformation. In particular there may now be island chains within island chains within island chains in the original J_i,φ_i coordinate system. As one repeats this process, the level curve diagram in the original J_2-plane becomes pathological indeed. Nonetheless Kolmogorov theory asserts that, for small energies most of the circular level curves of Hamiltonian (VI-1) are preserved; between the nondense set of preserved tori however there are highly complicated regions of instability which grow in size as the energy increases. The preceding method for identifying secondary resonances is not calculationally very useful. A simple method for identifying these secondary resonances comes from a straightforward application of the Birkhoff normalization procedure outlined in Chapter III. After one step in this normalization procedure has been performed, the resulting Hamiltonian will generally be an infinite series in the new variables. When expressed in terms of action-angle variables, many of these new terms will be resonances and each can be checked for its region of influence under the assumption that it acts alone. Moreover the legality of doing one or any finite number of steps in the Birkhoff normalization process is not open to question. Only when an infinite sequence of transformations are performed does the matter of convergence arise. It is true that in solving for the transformation equations by iterating one may have convergence only in a circle of finite radius. However, within the circle of convergence, the influence of resonances on the transformed Hamiltonian may be determined in the same way as were the influences of the primary resonances on Hamiltonian (VI-1).

This procedure furnishes a powerful technique for determining exactly which secondary resonance terms are important. We shall use it to search for low-order secondary resonances which appear at low energies in Hamiltonian (VI-1). We first use one step in the $\Omega_1 = \Omega_2$ modified Birkhoff technique to eliminate the 2-3 resonance. After the elimination, the new Hamiltonian is an infinite series. The seventh order resonant terms turn out to be proportional to $\cos(2\varphi_1 - 3\varphi_2)$, so that nothing of new interest is found here. There are no sixth or ninth order terms and no eighth order terms of interest. Another step in the Birkhoff procedure then eliminates the 2-3 resonant seventh order terms and gives rise to a ninth order 4-5 secondary resonance. The effects of this 4-5 secondary resonance should be observable in the original coordinates as a chain of five islands in the J_2-plane. The corresponding resonance zone should be in the neighborhood of the unperturbed 4-5 torus for the Hamiltonian (V-1). Using Equation (V-2), we see that $4\Omega_1 = 5\Omega_2$ provided

$$J_1 = \frac{1 + 22J_2}{7} \qquad \text{(VI-6)}$$

Since the smallest allowed value of J_2 in Equation (VI-6) is $J_2 = 0$, the minimum value for J_1 is 1/17. From Equation (V-1), we find that the minimum energy for the existence of this torus is 6/49. At any given energy, the unperturbed position of this torus can be found from Equation (V-1).

After these rather lengthy preliminaries, we now examine the level curves for Hamiltonian (VI-1) for energy $E = 0.20$, slightly below the 2-2, 2-3 overlap energy of 0.2095. The level curves in the J_2-plane are

shown in Figure 16. First we note that as before the stable periodic solutions due to the primary resonances are in approximately the correct positions. Most important, however, is the chain of five very small islands located on an approximate circle of radius 0.26 about the origin. The level curve for the unperturbed 4-5 torus is a circle of radius 0.268. Other chains of islands have been found, such as a chain of seven located on an approximate circle of radius 0.28. Large scale plots indicate that these islands are to computer accuracy smooth closed curves. The same is true for the other level curves in Figure 16.

A set of points obtained from an orbit close to the 2-2 separatrix is shown in Figure 17 for this energy. There is clearly evidence of minor breakdown in this curve. This is possibly due to some quite high order secondary resonance. Possibly if the integration were continued for a very long time with great accuracy, then one could see an island-like structure here. Thus we see that the transition from smooth level curves at some energies to a mixture of smooth curves and "ergodic" regions at higher energies, although relatively sharp, is not as sharp as indicated in the paper by Henon and Heiles. As a consequence, the simple method of finding the energy for which primary resonance zones, each acting independently, start to overlap can furnish only a reasonably accurate estimate for the occurrence of sizable regions where the smooth level curves have disappeared. The accuracy is, as we shall see in a future example, a function of the strength of the perturbations.

We have indicated previously the reasons for believing that the actual level curve diagram for Hamiltonian (VI-1) should be extremely complicated at all energies. Even at small energies there should be

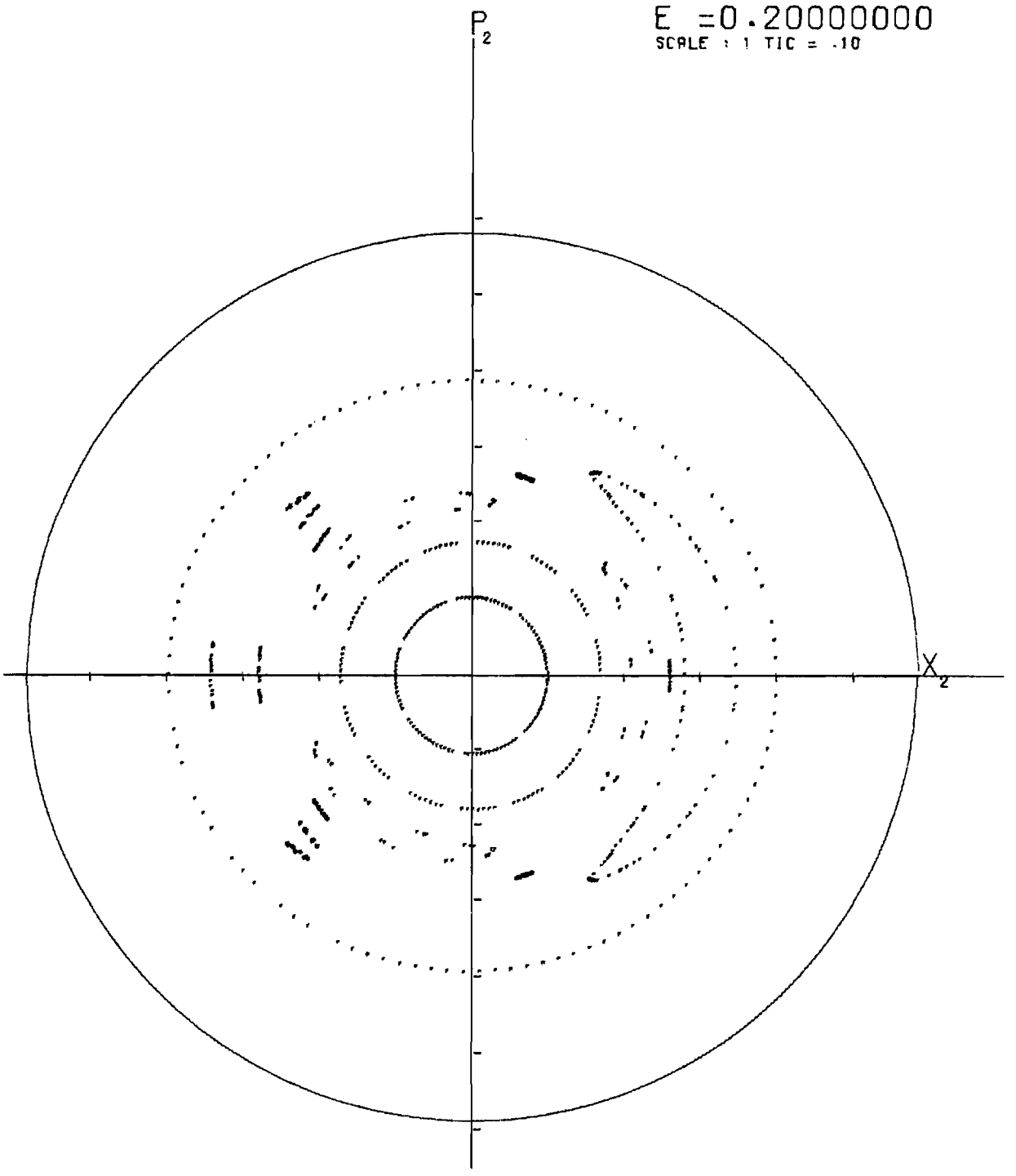

Figure 16. Level Curves for the First Model Hamiltonian.
Energy E = 0.20.

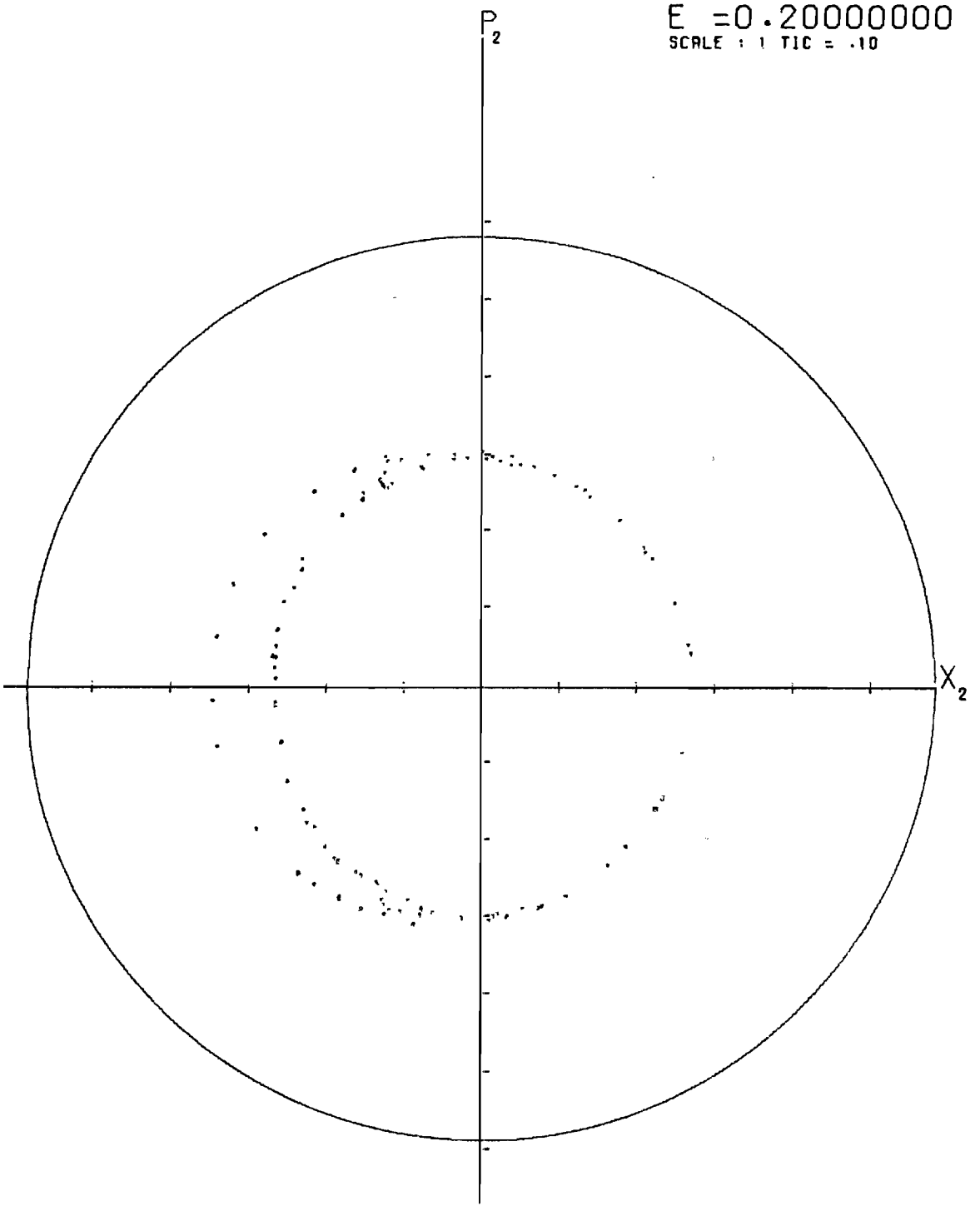

Figure 17. Level Curves for the First Model Hamiltonian.
Energy E = 0.20.

very thin high order island chains in most regions of phase space due to the effects of the secondary resonances. On a computer this "microscopic" structure is very difficult to detect, as the numerical integration of the equations of motion must be performed with extreme accuracy for long time intervals. Moreover there exist tori of the two integrable systems with α or $\beta = 0$ which are slightly distorted for α and β both nonzero. Only in gaps between these preserved tori does one find regions where the secondary resonances produce the complicated island structure mentioned above. These preserved tori exert a stabilizing influence on the appearance of the level curves; for since the gap between successive preserved tori is small for small energies, a complicated island structure or an ergodic orbit in this gap would still to computer accuracy yield a simple level curve. It is only when these preserved tori become widely separated, such as when resonance zones overlap, that zones of instability become macroscopic for example when we cause the primary 2-2 and 2-3 resonance zones to overlap as in Figure 15. The macroscopic effects due to the overlap of the secondary resonances are visible in Figure 17.

In summary, we observe that determining the energy of the overlap for the primary resonances gives a good approximation to the energy of appearance of macroscopic irregularities in the behavior of dynamical systems with two degrees of freedom. We now review the method for improving the accuracy of predicting when this breakdown will become visible. A practical method consists of performing several stages in the Birkhoff normalization to identify the low-order secondary resonances. The limit on the order to which diagonalization should be carried is a function of the accuracy of the integration technique used

to find the level curves. That is, if the integration methods used cannot resolve a chain of twenty islands, for example, then the breakdown caused by a 20-21 resonance would not normally be visible. Once the proper secondary resonance has been identified, then it is an easy matter to calculate the position of the unperturbed torus having the proper frequencies to be broken down by this resonance. The resonance zone for this term will certainly have a finite width, but this is generally so small that it is negligible. Whenever this torus intersects one of the primary resonance zones however, a visible irregularity of the level curves may result. We shall apply this method to another example.

It would be of considerable interest to look at the level curves for Hamiltonian (VI-1) for higher energies to see if the regions with seemingly regular level curves shrink as did the level curves for the Henon and Heiles system. Unfortunately the mean period of oscillation for Hamiltonian (VI-1) increases with energy making the integration time necessary to obtain level curves prohibitively long. For that reason, as well as to apply our method to a system with larger perturbations, we shall now study the Hamiltonian

$$H = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2 + \alpha J_1 J_2 \cos 2(\varphi_1 - \varphi_2) + \beta J_1^{3/2} J_2 \cos(3\varphi_1 - 2\varphi_2). \tag{VI-7}$$

If $\beta = 0$, this is the familiar system described by Equation (V-10). For $\alpha = 0$ this is Hamiltonian (V-15) which, as we saw in Chapter V, has a chain of three islands in the J_1-plane. For this reason we shall investigate the level curves in the J_1-plane. The parameters, α and β, will be fixed throughout at $\alpha = 0.95$ and $\beta = 0.25$.

As in the previous example we shall use the positions of the separatrices for the integrable $\alpha = 0$ or $\beta = 0$ systems, in the J_1-plane as a measure of the proximity of the resonance zones. The inner point of intersection of the 2-2 separatrix with the positive x_1-axis can be found numerically from Equations (V-11) and (V-13). The outer point of intersection of the 3-2 separatrix with the positive x_1 axis is determined from Equations (V-17) and (V-19). These distances as functions of energy are shown in Figure 18. In Chapter V we found that for Hamiltonian (V-15) the chain of three islands was absent until $E = 14/169$. Thus there will be an energy range for which overlap of the primary resonance zones has not occurred. Finally we note from Figure 18 that the energy of overlap for the 2-2 and 3-2 resonance zones is approximately 0-12.

At $E = 0.05$, the level curves in the J_1-plane for Hamiltonian (VI-7), shown in Figure 19, seem to be regular. However, even at this low energy, a high order island chain has been found by doing a careful numerical study of the level curves. The initial conditions for this orbit are $x_1 = 0.0797011$, $p_1 = 0.013139$, $x_2 = 0$, and $p_2 = 0.30091909$. This five island chain, not shown in Figure 19, is due to a 5-4 secondary resonance. Under normal graphical accuracy this island chain appears to be an ordinary "circular" level curve. However, by increasing the scale used in plotting by a factor of 100 the island-like appearance is resolved. This example provides a confirmation of the validity of our previously expressed view concerning the level curves for a multiply resonant system. Using two steps in the Birkhoff normalization, it may be verified that there is indeed a 5-4 secondary resonance in Hamiltonian

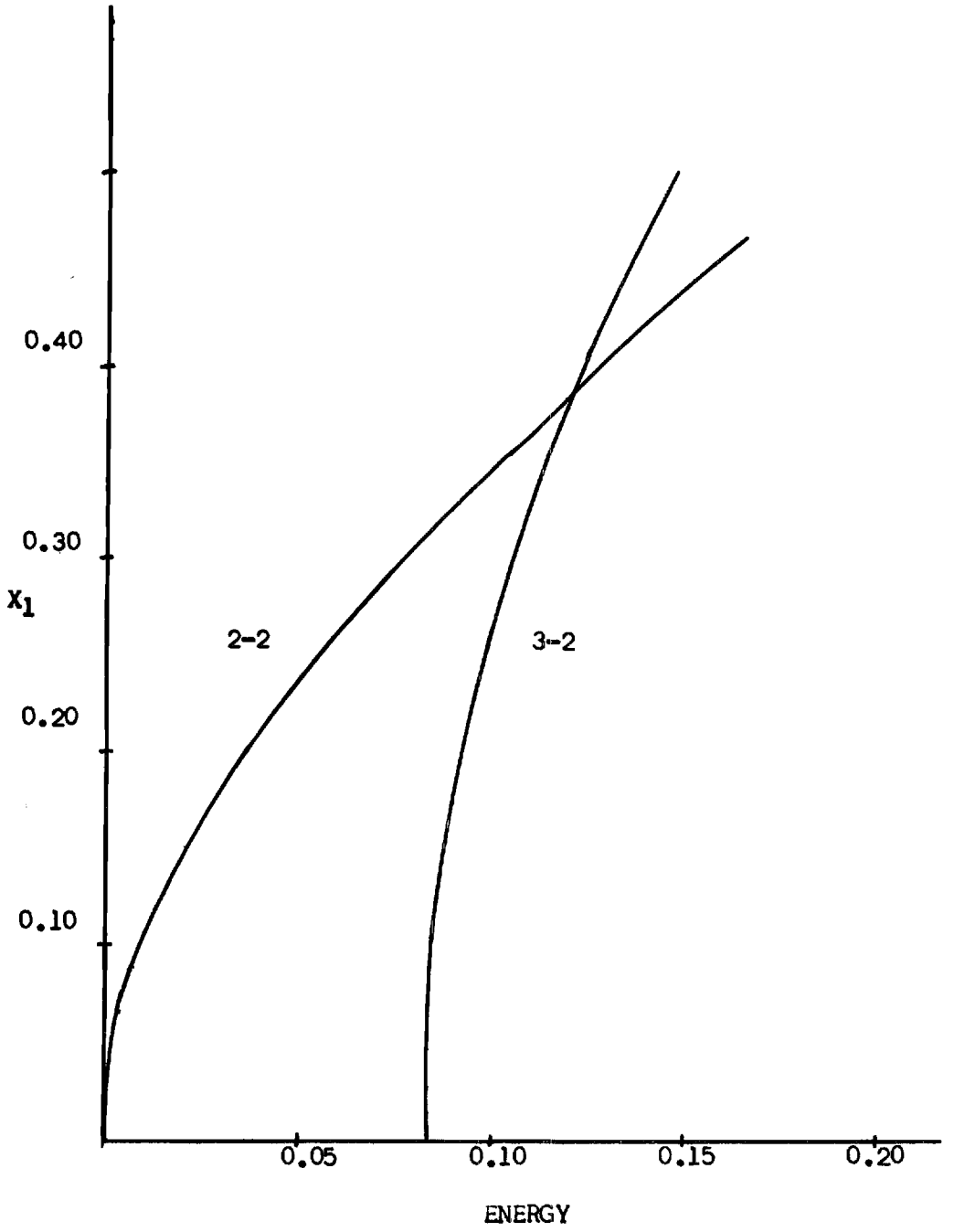

Figure 18. Position of the 2-2 and 3-2 Separatrices as a Function of Energy.

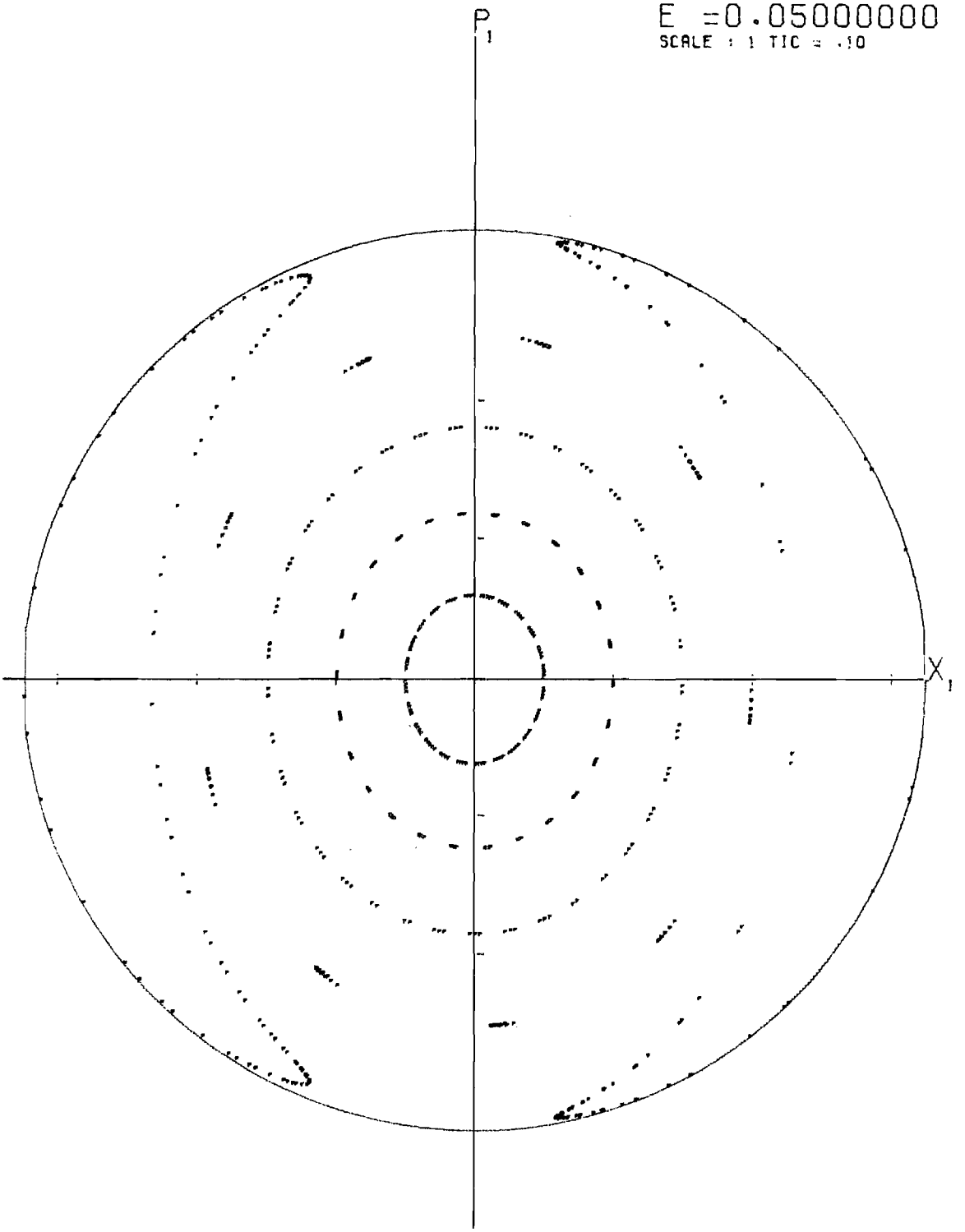

Figure 19. Level Curves for the Second Model Hamiltonian.
Energy E = 0.05.

(VI-7). For the unperturbed Hamiltonian (V-1) the location of the 5-4 torus is found by setting $5\Omega_1 = 4\Omega_2$ and using Equation (V-2). Thus on the unperturbed 5-4 torus

$$J_2 = \frac{1 + 2J_1}{23} . \qquad \text{(VI-8)}$$

We find from Equations (VI-8) and (V-1) that at E = 0.05, the level curve for this torus is a circle of radius 0.098. The actual position of the island chain at this energy was on a circle of radius 0.08.

We next consider the level curves for this system at energy E = 0.08. As is evident from Figure 20, the chain of islands from the 5-4 resonance is now macroscopic in size. There is also clear evidence of breakdown near the 2-2 separatrix. Other level curves not shown in Figure 20 confirm this. A chain of nine islands has been found intersecting the x_1 axis at around $x_1 \cong 0.30$ which is very close to the 2-2 separatrix. This is due to a 9-8 secondary resonances in at least seventeenth order. This secondary resonance or perhaps one of even higher order is the cause of breakdown at this energy. In principle at least it should be possible to use the Birkhoff normalization to isolate the secondary resonance causing breakdown and to predict the energy as we did in the previous example, although we shall not enter into this calculation here. One should especially note that at this energy the primary chain of three islands has not yet appeared; hence for these relatively large perturbations, it is essential to use the method of secondary resonances.

In Figures 21 and 22 the level curves for Hamiltonian (VI-17) are

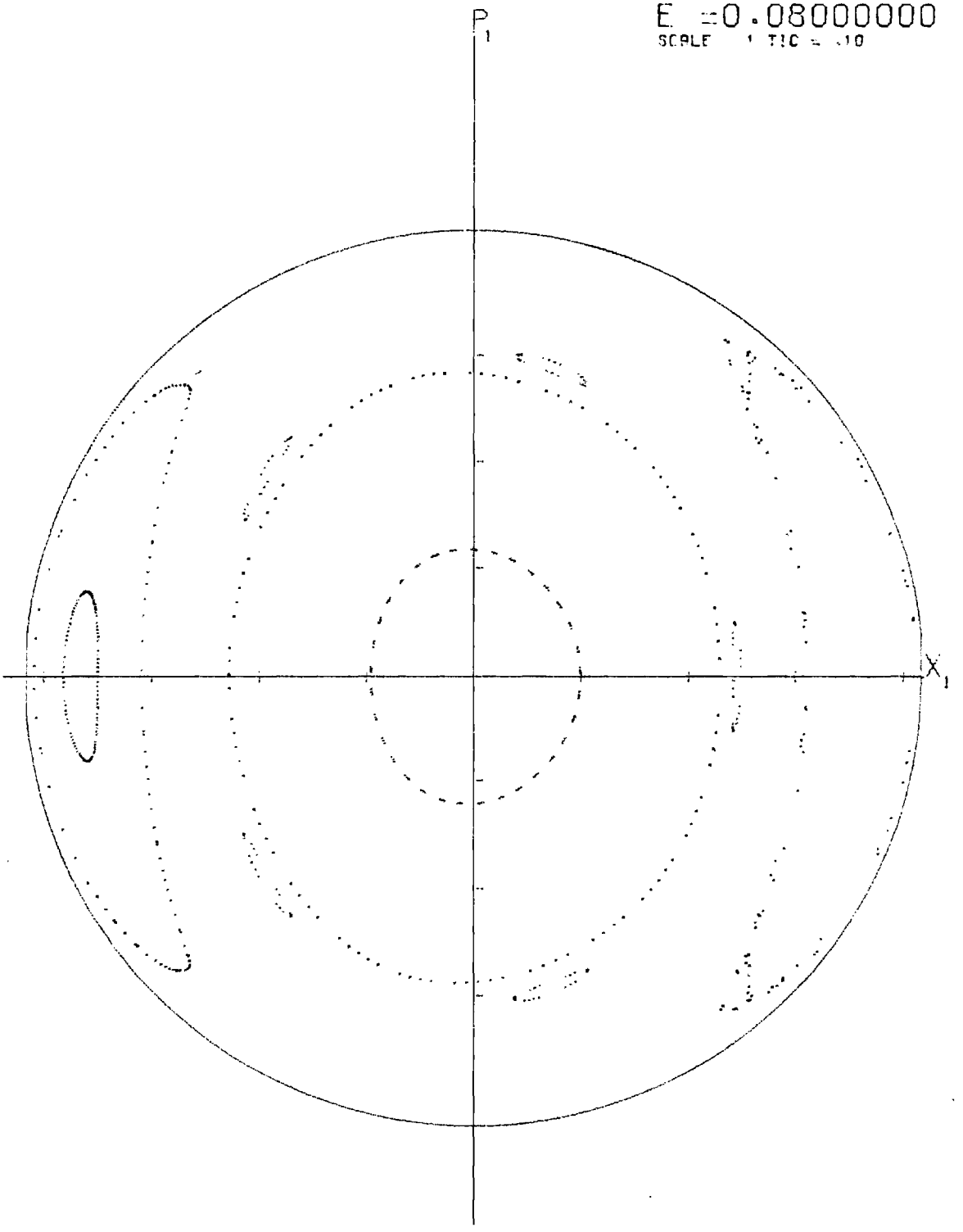

Figure 20. Level Curves for the Second Model Hamiltonian.
Energy E = 0.08.

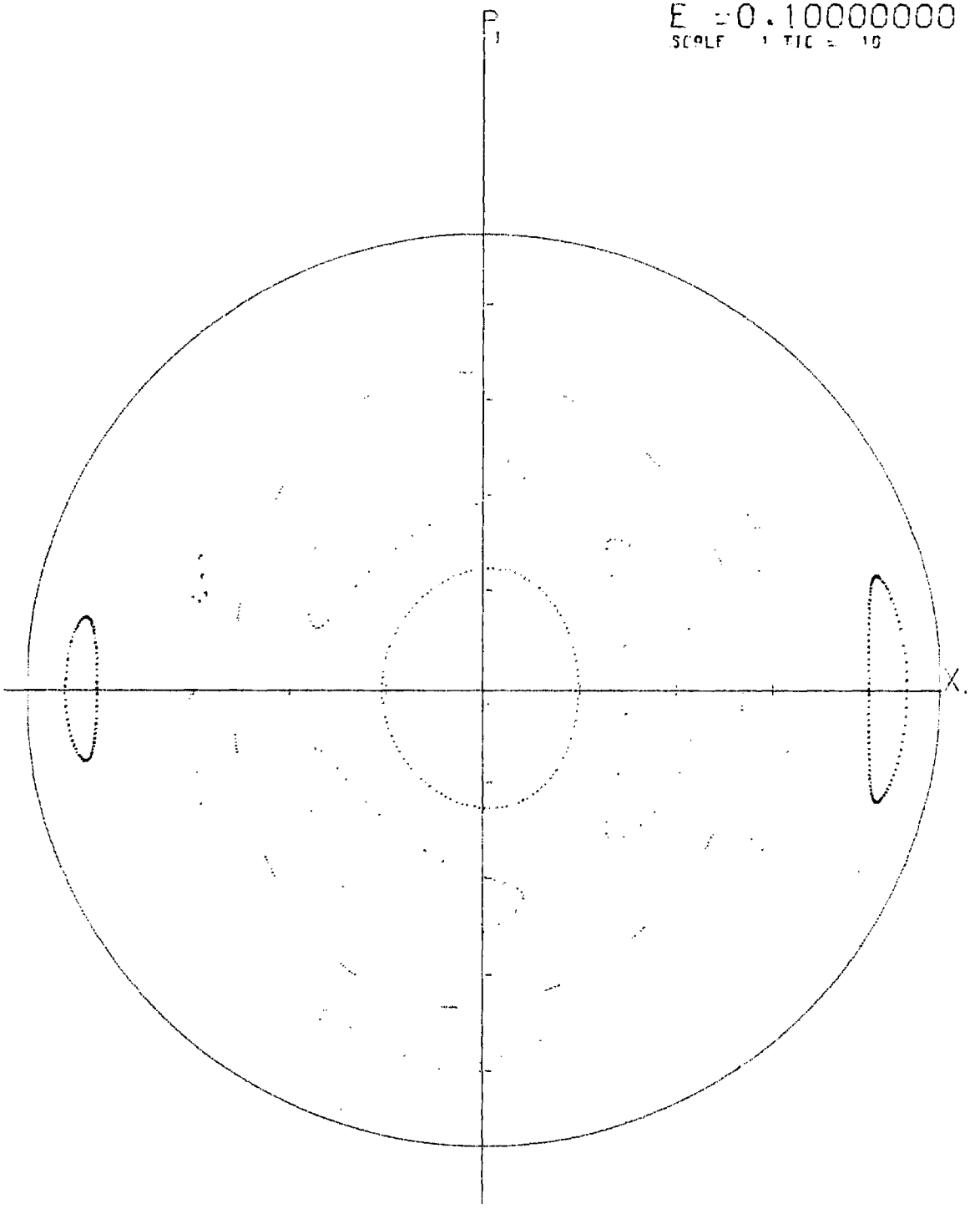

Figure 21. Level Curves for the Second Model Hamiltonian.
Energy E = 0.10.

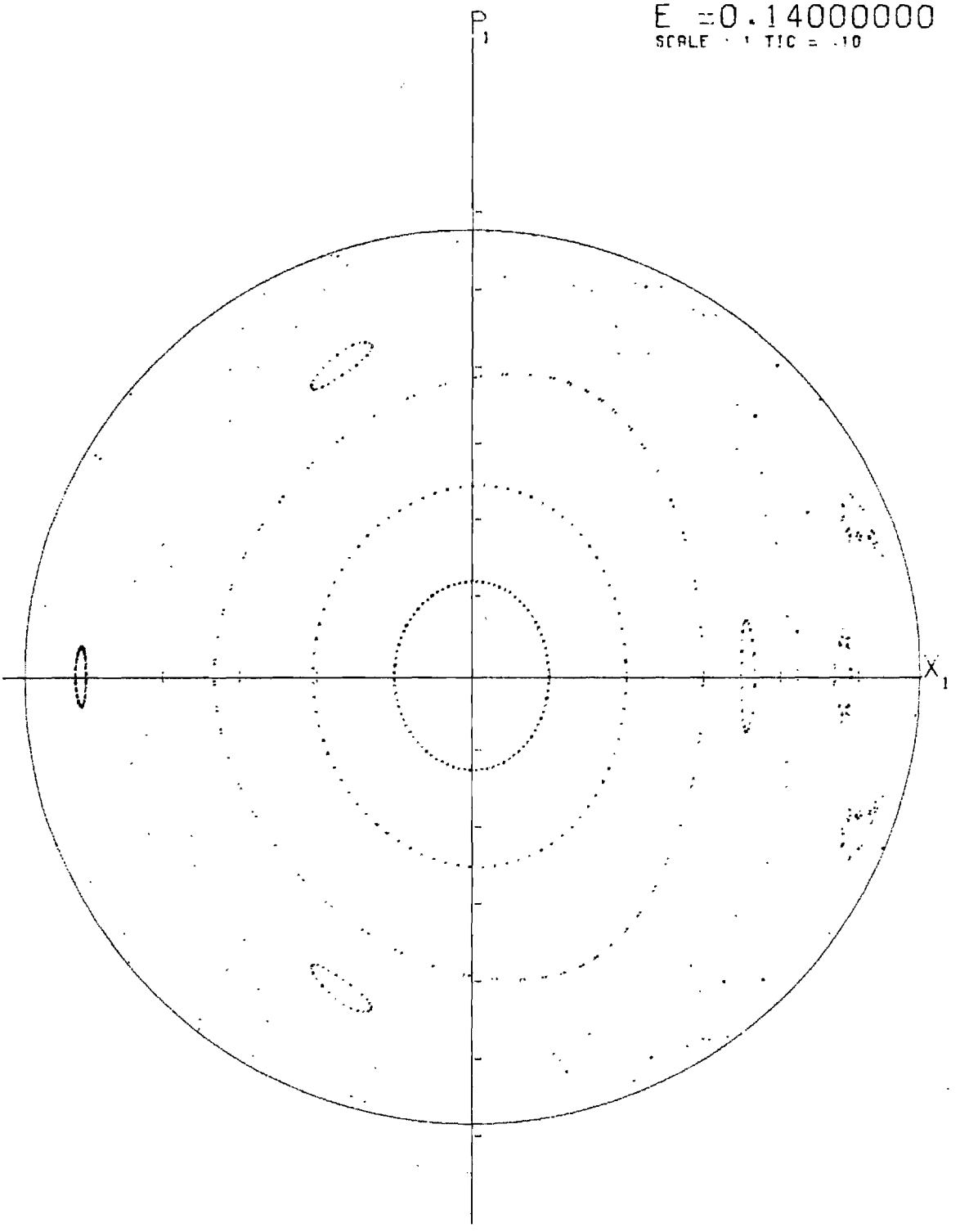

Figure 22. Level Curves for the Second Model Hamiltonian. Energy E = 0.14.

shown for energies E = 0.10 and E = 0.14 respectively. We note that the regions containing no closed level curves increase with increasing energy. The behavior here is analogous to that of the Henon and Heiles system. In the next chapter we shall discuss the Henon-Heiles result in light of the work of the last three chapters.

CHAPTER VII

CONCLUSION

In the previous three chapters we have investigated the consequences of perturbing an integrable Hamiltonian and we have isolated those crucial terms in the perturbation, the resonances, which cause a marked change in the system behavior regardless of the strength of the perturbing terms. First, starting with a Hamiltonian which was a function of the action variables alone, it was shown that the addition of a single resonance as a perturbation produced a dynamical system that was still integrable. We then saw that the level curves for these integrable systems could have chains of islands similar to those found in the work of Henon and Heiles. Furthermore, we exhibited simple algebraic schemes for finding the location of these island chains in level curve diagrams. Finally it was shown by two examples that a Hamiltonian with two or more distinct resonances can, when the resonant zones overlap, yield a breakdown in level curve structure similar to that exhibited by the Henon and Heiles system. Moreover, straightforward methods were established for predicting the region and energy of this breakdown. We now return to the Henon and Heiles potential given in Equation (II-15) and try to relate these results to the observed level curve features for that system.

One of the major qualitative differences between the level curves for the two model problems of Chapter VI and the level curves for Hamiltonian (II-15) is the fashion in which most of the island chains appear.

In the two model problems, Hamiltonians (VI-1) and (VI-7), the island chains which were analytically investigated appear at the origin in one coordinate plane and move outward as the energy increases while in the other coordinate plane they appear at the bounding level curve. Almost all island chains that were found for these two systems, many of them not shown in the figures, lay along distorted circles centered on the origin. On the other hand, in the Henon-Heiles system all the island chains first appear about stable periodic solutions which do not lie at the origin. An example of this is the chain of five islands in Figure 4. We shall now examine one of the model problems, Hamiltonian (VI-7), in order to establish that this difference does not have any fundamental significance.

Hamiltonian (VI-7) has two anomalous periodic solutions where first $J_1 = 0$ and then $J_2 = 0$. In the J_1-plane these orbits are represented by a point at the origin and by the whole bounding level curve. In fact the bounding level curve includes every point of an entire orbit. Since there are tori supporting conditionally periodic motion arbitrarily close to each of these orbits, the orbits are stable. In a sense then, this model system has the property that the island chains first appear about stable periodic solutions just as do the islands in the Henon-Heiles system. However the similarity between the two systems is obscured because the stable periodic solutions for the model Hamiltonian appear either at the origin or the bounding curve. There are a number of canonical transformations which can be used to move the $J_1 = 0$ and $J_2 = 0$ solutions from their somewhat unique positions and which serve to show that the model problems are quite similar to the Henon-Heiles system. The simplest

canonical transformation that will do this is an orthogonal transformation to new variables, u_i, P_i, defined by

$$u_1 = \frac{x_2 + x_1}{\sqrt{2}} , \qquad \text{(VII-1a)}$$

$$u_2 = \frac{x_2 - x_1}{\sqrt{2}} , \qquad \text{(VII-1b)}$$

and

$$P_1 = \frac{p_2 + p_1}{\sqrt{2}} , \qquad \text{(VII-2a)}$$

$$P_2 = \frac{p_2 - p_1}{\sqrt{2}} , \qquad \text{(VII-2b)}$$

This transformation represents a rotation by 45 degrees in the $x_1 - x_2$ plane. It is convenient to introduce new action-angle variables, $\mathcal{J}_i, \theta_i$, defined by

$$u_i = \sqrt{2\,\mathcal{J}_i}\,\cos\theta_i , \qquad \text{(VII-3a)}$$

$$P_i = -\sqrt{2\,\mathcal{J}_i}\,\sin\theta_i ; \; i = 1, 2. \qquad \text{(VII-3b)}$$

The relations between the original action-angle variables, J_i, φ_i, and the new action-angle variables, $\mathcal{J}_i, \theta_i$, are, from Equations (II-3), (VII-1), (VII-2) and (VII-3),

$$J_1 = \tfrac{1}{2}\left(\mathcal{J}_1 + \mathcal{J}_2 - 2\sqrt{\mathcal{J}_1 \mathcal{J}_2}\,\cos(\theta_1 - \theta_2)\right), \qquad \text{(VII-4a)}$$

$$J_2 = \tfrac{1}{2}(\mathcal{J}_1 + \mathcal{J}_2 + 2\sqrt{\mathcal{J}_1\mathcal{J}_2}\cos(\theta_1-\theta_2)). \qquad \text{(VII-4b)}$$

Using Equations (VII-4) and denoting the angle independent terms in Hamiltonian (VI-7) by H_o, we have

$$H_o = \mathcal{J}_1 + \mathcal{J}_2 - \tfrac{3}{4}(\mathcal{J}_1 + \mathcal{J}_2)^2 + 2(\mathcal{J}_1 + \mathcal{J}_2)\sqrt{\mathcal{J}_1\mathcal{J}_2}\cos(\theta_1-\theta_2) + 3\mathcal{J}_1\mathcal{J}_2\cos^2(\theta_1-\theta_2) \qquad \text{(VII-4b)}$$

as the unperturbed Hamiltonian in the rotated coordinates.

It is straightforward to find the level curves of Hamiltonian (VII-5). In the $\mathcal{J}_1$ (or u_1, P_1) plane the level curves lie inside a bounding circle centered on the origin as shown in Figure 23. Almost all the level curves in Figure 23 are ovals about the two stable periodic solutions lying on the P_1 axis. Equation (VII-4) implies that the level curve on the positive P_1 axis, consisting of a single point, is the image of the same orbit that gave the single point, level curve at the origin in the J_1-plane. Similarly, one can show that the old bounding level curve in the J_1-plane is transformed into the single point, level curve on the negative P_1 axis. In addition, there is now a "dividing" curve on the u_1 axis. This line, however, is not a separatrix. Thus the effect of the coordinate rotation is to transform the former unperturbed circular level curves into two sets of ovals centered on stable periodic solutions which were previously the origin and bounding level curve. The level curves for rotated, unperturbed Hamiltonian (VII-5) are now almost identical, except for a ninety degree rotation, to those of the Henon-Heiles system which are shown for E = 0.01 in Figure 3 of the article by

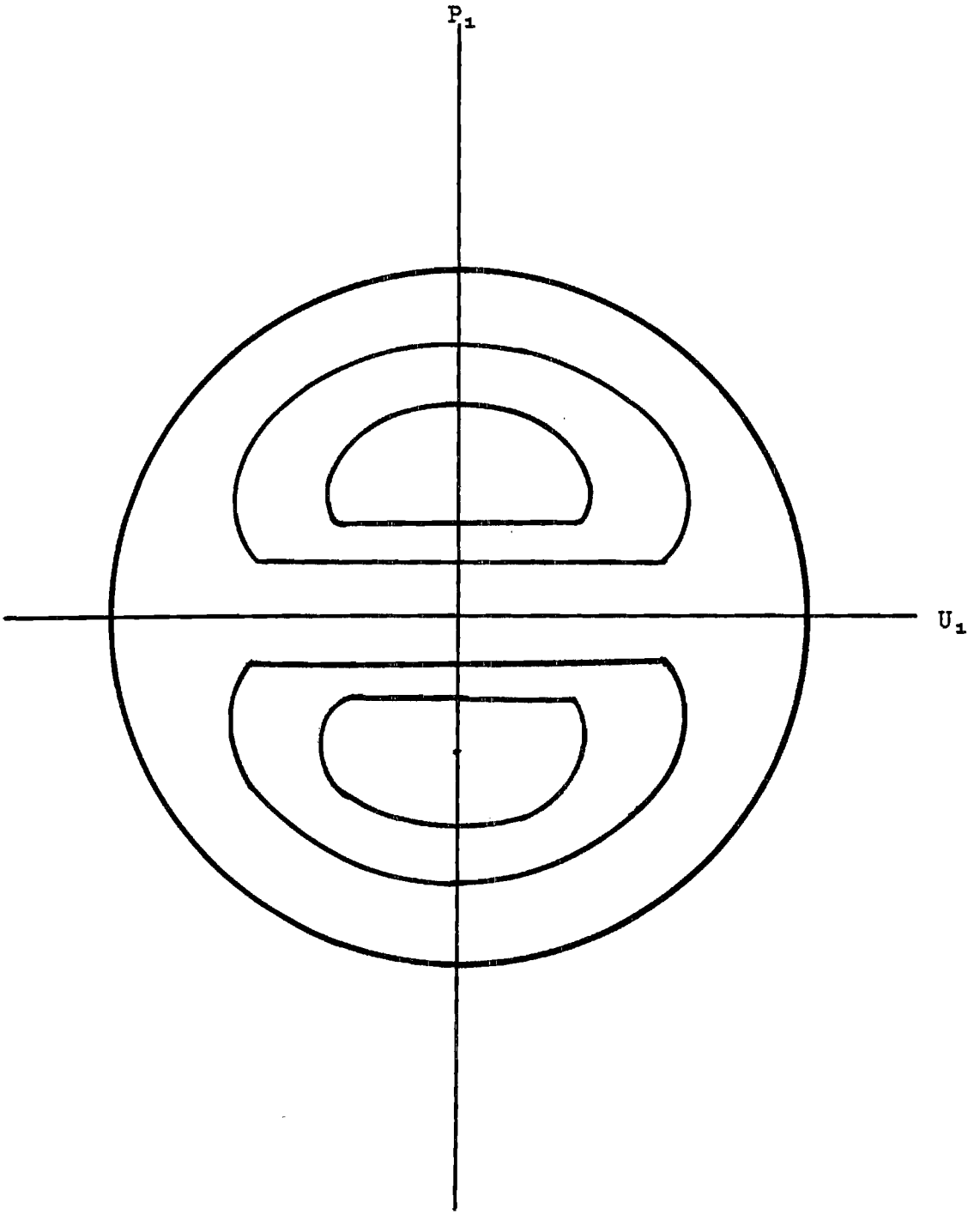

Figure 23. Level Curves for $H = J_1 + J_2 - J_1^2 - 3J_1J_2 + J_2^2$ in the Rotated Coordinate System (not to scale).

Gustavson (Gustavson 1966). Moreover, if two stages in the Birkhoff normalization are performed on the Henon-Heiles Hamiltonian and the resulting Hamiltonian truncated at fourth order, one obtains

$$H = J_1 + J_2 - \frac{5}{12} J_1^2 + \frac{3}{2} J_1 J_2 - \frac{5}{12} J_2^2 - \frac{7}{3} J_1 J_2 \cos^2(\varphi_1 - \varphi_2). \quad \text{(VII-6)}$$

Hamiltonians (VII-5) and (VII-6) are so similar that one is no longer surprised that they lead to similar level curves. Consequently, it becomes clear that our model Hamiltonians are in a sense only one canonical transformation away from the Henon-Heiles system. Moreover, it is possible to investigate the effects of 2-2 and 3-2 resonance perturbations acting on the unperturbed rotated model Hamiltonian (VII-5) using the methods of the last chapter, i.e., studying the effects of the 2-2 and 3-2 resonances individually. However, the algebra is rather cumbersome and we will not reproduce the details of this study. Instead we shall proceed directly to the level curves of the rotated version of Hamiltonian (VI-7). In the $\mathcal{J}_1$-plane these level curves are greatly distorted versions of the original level curves for this system. A typical set of level curves are shown in Figure 24 for energy $E = 0.09$, $\alpha = 0.95$, and $\beta = 0.25$. Figure 24 should be compared with Figure 21 which shows the unrotated level curves. The 2-2 separatrix has been shifted to the lower half-plane and is broken down to some extent. The chain of three islands from the 3-2 resonance is clearly visible about the stable periodic solution in the upper half-plane. Clearly the level curves of Figure 24 exhibit all of the features for the Henon-Heiles and related systems.

We see therefore that there are no essential qualitative differences

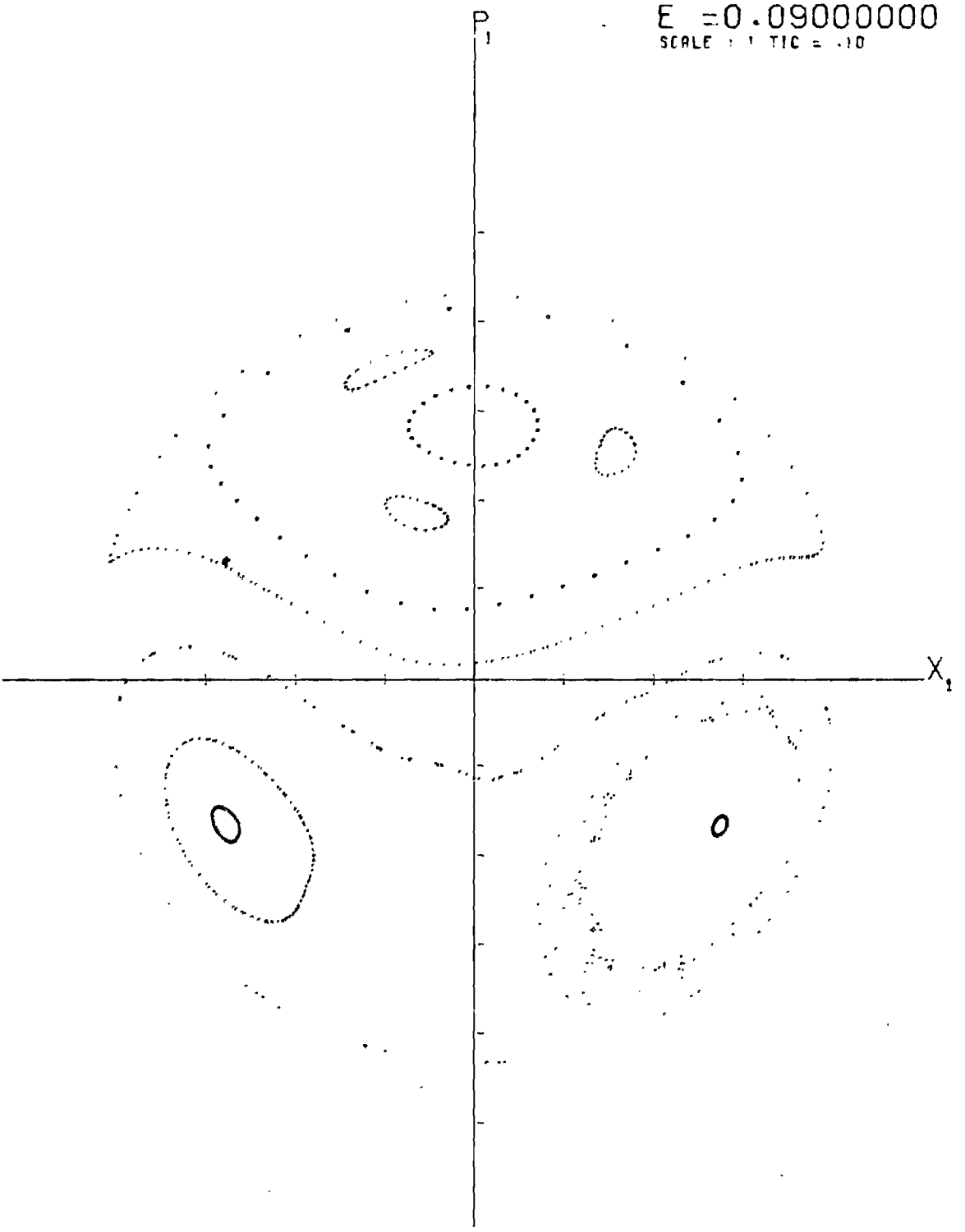

Figure 24. Level Curves for the Second Model Hamiltonian in the Rotated Coordinate System. Energy E = 0.09.

in the behavior of the two model Hamiltonians and the physically more realistic potential used by Henon and Heiles. Consequently it should be possible to apply the methods of Chapter VI to obtain a reasonably accurate estimate for the energy at which breakdown in the Henon-Heiles level curves should become apparent. For the Henon-Heiles system there are a number of non-essential but algebraically complicating factors that do not occur in the model Hamiltonians. First, the Henon-Heiles Hamiltonian given by Equation (II-15), when expressed in terms of the action-angle variables of Equation (II-3) becomes

$$H = J_1 + J_2 + J_1\sqrt{\frac{J_2}{2}}\cos(2\varphi_1+\varphi_2) + J_1\sqrt{2J_2}\cos\varphi_2 \qquad \text{(VII-7)}$$
$$+ J_1\sqrt{\frac{J_2}{2}}\cos(2\varphi_1-\varphi_2) - \frac{1}{3\sqrt{2}}J_2^{3/2}\cos 3\varphi_2 - \frac{1}{\sqrt{2}}J_2^{3/2}\cos\varphi_2.$$

The methods of Chapter VI do not apply to this Hamiltonian as it stands since there are angle dependent nonresonant terms present. However it is possible to remove these nonresonant terms by using the Birkhoff normalization method of Chapter III. The goal here is not to reduce the Hamiltonian to an integrable form, but simply to exhibit all the latent resonances. Once the nonresonant terms are eliminated, one must then eliminate the dominant, low order 2-2 resonant terms, converting them to pure action terms using the techniques mentioned around Equation (VI-3). In essence one needs to transform the Henon-Heiles system, which is analogous to the "rotated" Hamiltonian (VII-5), to a form analogous to the model Hamiltonians of Chapter VI. Whittaker (Whittaker 1961) guarantees that such a transformation exists though it is certainly more complicated

than transformation (VII-1,2). Once nonresonant and low order 2-2 resonant terms have been eliminated, the Henon-Heiles system may be treated according to the methods of Chapter VI. We have not carried out this procedure in this thesis since the essentials of breakdown can be illustrated without entering into the algebraic complications of the Henon-Heiles system.

The complete reduction of Hamiltonian (VII-7) to a multiply resonant form similar to those of Chapter VI would involve an infinite sequence of canonical transformations. In practice this process could be terminated at some relatively low order. If the basic problem is to determine the region and energy of the appearance of irregularities in the level curves, then one need not estimate breakdown to a greater accuracy than could be observed on the level curve diagrams. Breakdown regions smaller than those observable within the limits of accuracy used in plotting level curves could not be detected. For example, if level curve plots permit the resolution of a chain of twenty islands, but none higher, than it is unlikely that the effects of terms in the Hamiltonian of degree greater than around forty five could be detected. Once Hamiltonian (VII-7) has been reduced to a multiply resonant form through the desired order, it is possible in principle to find algebraically the width and location of the resonance zones using the methods of Chapter VI. At very low energies the resonance zones would not overlap and the level curves would appear to be regular. At higher energies macroscopic overlap would occur, and irregularities in the structure of the level curves would be evident. One could find, in principle, the energy for which breakdown first occurs using the methods of Chapter VI.

Having established the origin of amplitude instability in nonlinear systems as being due to the overlap of resonances and having shown that one can predict the energy for which amplitude instability becomes apparent in computer calculations, let us now return to the question concerning the existence of an isolating third integral or equivalently to the integrability of Hamiltonian (II-5). In Chapter III, we showed that one could calculate a formal third integral, and in Chapter IV we demonstrated by example that this formal integral correctly predicted the level curves for energies small compared to the dissociation energy. Despite this fact, Kolmogorov theory assures us that the formal integral of Chapter III cannot correctly predict the level curves in the very small zones of instability caused by overlapping resonances which occur even when the energy is small. Indeed even the computer calculated level curves begin to show these zones of instability as the energy increases. Nonetheless, for small energy, the formal third integral is valid for most initial conditions; therefore from the physicist's viewpoint, though not the mathematician's, nonlinear systems undergoing small amplitude motion are integrable.

When the zones of instability become sufficiently large to be computer observable, the question concerning the existence of a useful third integral becomes debatable. The computer indicates smooth level curves associated with island chains even in the zones of instability. It is certainly possible that one could, as suggested by Jefferys (Jefferys 1966), calculate an expansion for a third integral which would be valid in certain restricted regions containing even complicated island chains. However, as the energy increases and as the island chains begin

to contain many islands spread over a wide region of the level curve plane, a highly complicated mathematically isolating integral would not be deserving of that name from the physicist's viewpoint. Certainly a computer cannot be used to resolve the mathematical question of isolating versus nonisolating integral as the energy approaches dissociation. Nonetheless, when the computer calculated level curve plot shows the widespread amplitude instability of Figure 5, physically speaking the system does not possess an isolating third integral and is "nonintegrable" or ergodic. This study then has been devoted to predicting the onset of "physical ergodicity."

Finally let us now return to the question posed in Chapter I concerning the usefulness of nonlinear models in physics. All results for systems of two oscillators indicate that "physical ergodicity" becomes significant only for amplitudes approaching those of dissociation. Thus amplitude instability could be quite significant for chemical molecules undergoing dissociation or solids near the melting point. However, unless the energy at which amplitude instability occurs decreases as the number of oscillators increases, amplitude instability plays an insignificant role in making nonlinear systems useful in physics. Clarification of this point is left for future study.

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